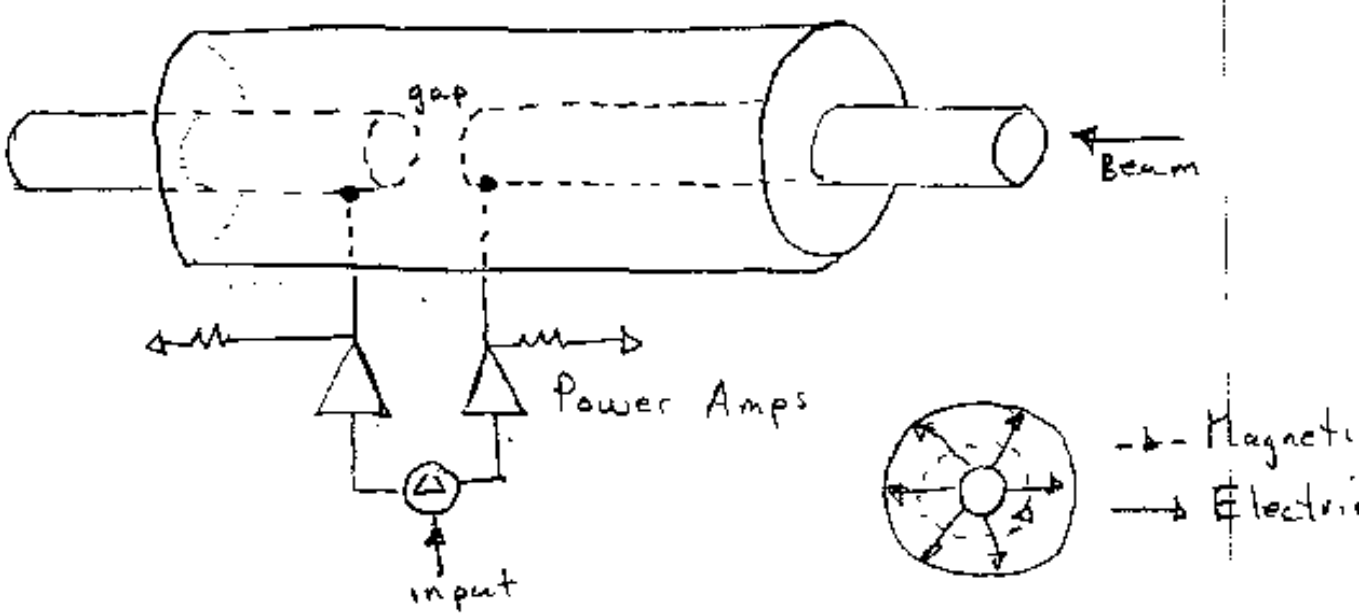
Linear Circuits

Reference: Engineering Circuit Analysis
W. H. Hayt & J. E. Kemmerly

Typical RF cavity circuit.

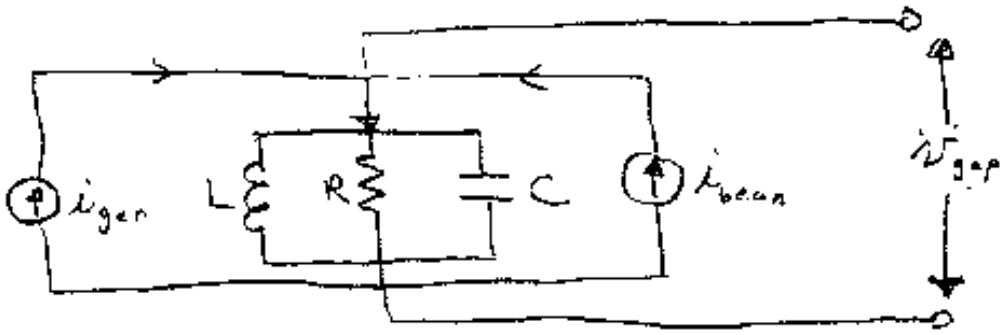
Single gap coax cavity



Stored Electric fields $\Rightarrow$ Capacitance $q = Cv$

Stored Magnetic fields $\Rightarrow$ Inductance $\Phi = Li$

Equivalent Circuit



$$v_{gap} = L \frac{d\, i_{ind}}{dt} \qquad \frac{1}{L} \int V_{gap}\, dt = i_{ind}$$

$$\frac{v_{gap}}{R} = i_{res}$$

$$C \frac{dv_{gap}}{dt} = i_{cap}$$

$$C \frac{dv}{dt} + \frac{1}{R} v + \frac{1}{L} \int v\, dt = i_{gen} + i_{beam}$$

Homogeneous solution. (transient or natural response)

Total Input current $= i_{gen} + i_{beam} + 0$

Homogeneous response due to initial conditions such as initial charge on capacitor and initial flux in inductor

$$\frac{dv}{dt} + \frac{1}{RC} v + \frac{1}{LC} \int v\, dt = 0$$

Solution is of the form $v(t) = V(s) e^{st}$

$$\frac{V(s)}{s} \left(s^2 + \frac{1}{RC} s + \frac{1}{LC} \right) = 0$$

Non-trivial solution

$$s^2 + \frac{1}{RC} s + \frac{1}{LC} = 0$$

<u>Characteristic Equation</u>

$$s = -\left(\frac{1}{2RC}\right) \pm j\left(\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2\right)^{1/2}$$

Define $\omega_0 = \frac{1}{\sqrt{LC}}$ (rad/sec)

$$Q = \omega_0 RC$$

$$s = -\frac{\omega_0}{2Q} \pm j\left(\omega_0^2 - \left(\frac{\omega_0}{2Q}\right)^2\right)^{1/2}$$

$$v(t) = a_1 e^{-\left(\frac{\omega_0}{2Q}\right)t} e^{+j\left(\omega_0^2 - \left(\frac{\omega_0}{2Q}\right)^2\right)^{1/2} t}$$

$$+ a_2 e^{-\frac{\omega_0}{2Q}t} e^{-j\left(\omega_0^2 - \left(\frac{\omega_0}{2Q}\right)^2\right)^{1/2} t}$$

where a_1 & a_2 are determined from initial condition

or:

$$v(t) = V_0 e^{-\frac{\omega_0}{2Q}t} \cos\left(\left(\omega_0^2 - \left(\frac{\omega_0}{2Q}\right)^2\right)^{1/2} t + \delta\right)$$

where V_0 and δ are determined from initial conditions

Roots of the Characteristic Eqn are in general complex numbers

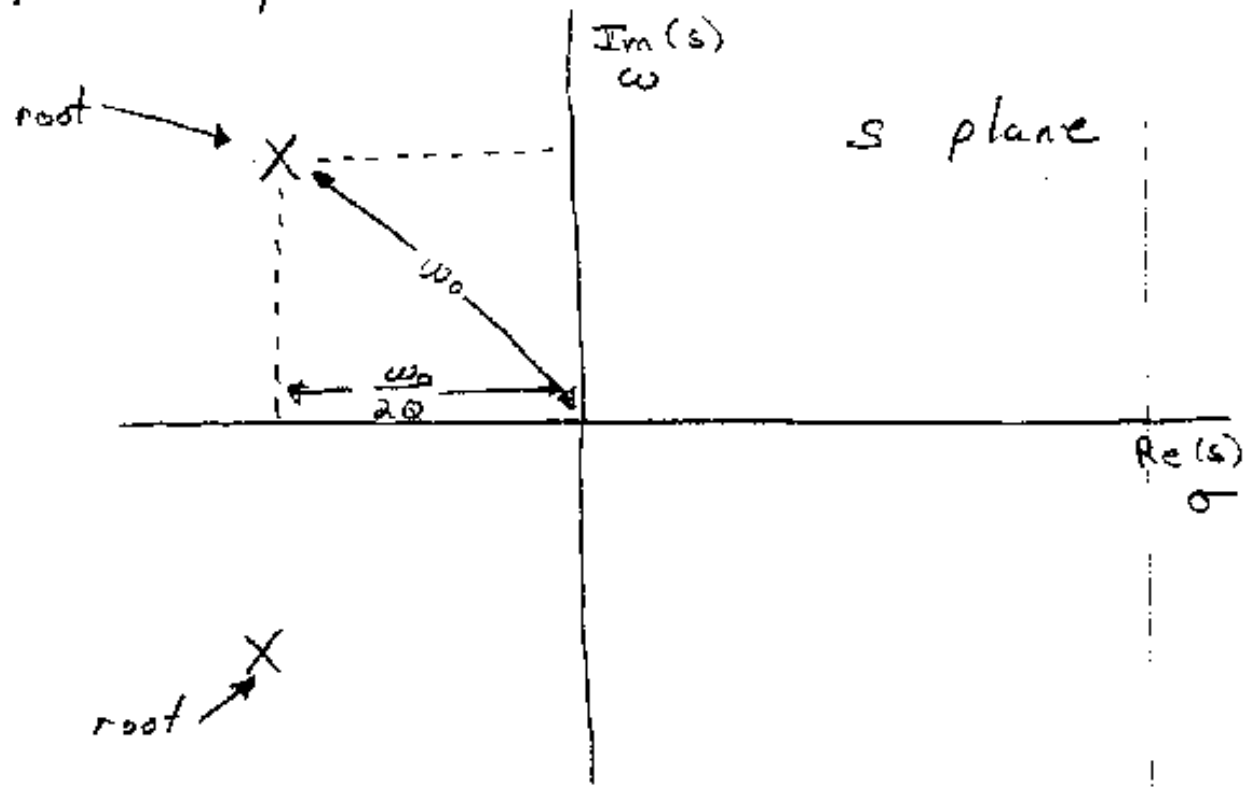
$$S_{root} = \sigma + j\omega$$

Decaying part

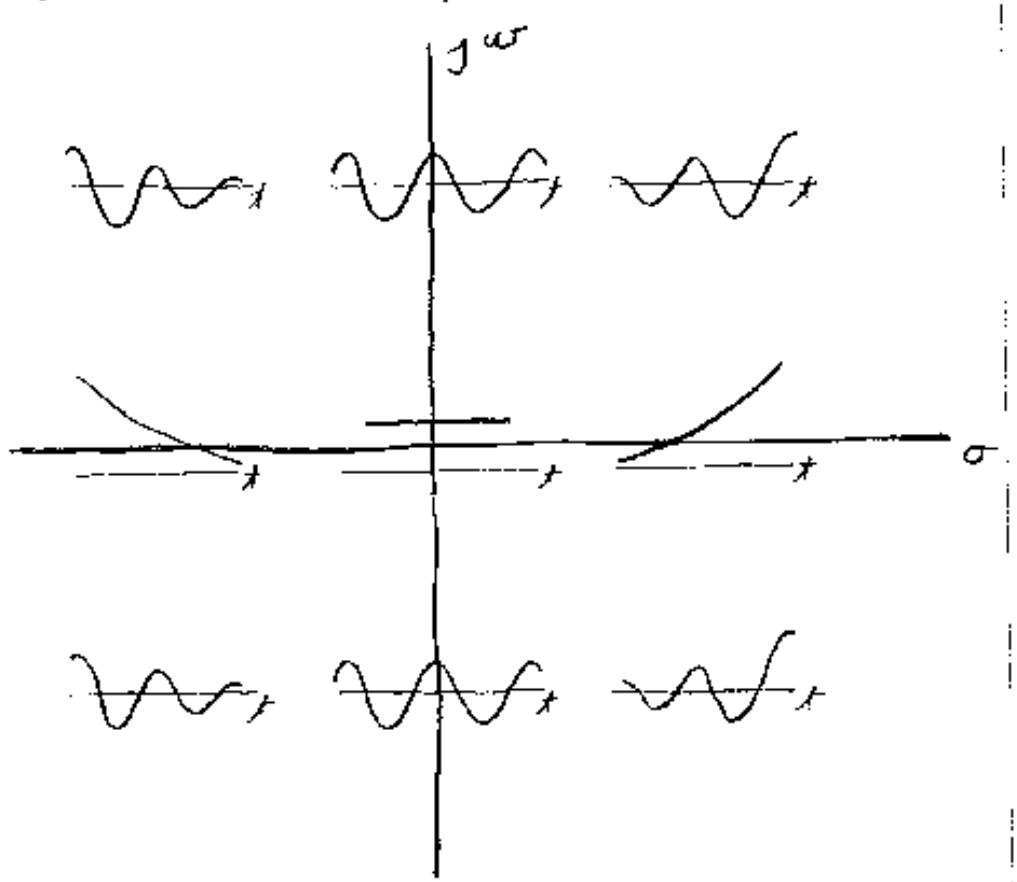
Oscillating part

We can map the roots out in the s-plane

For the example RLC circuit above:



In general, the roots may lie anywhere on the s-plane.



The Forced Response

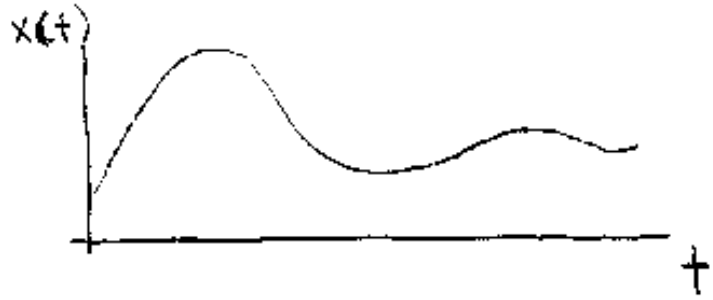
$$x(t) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(t)$$

Linearity

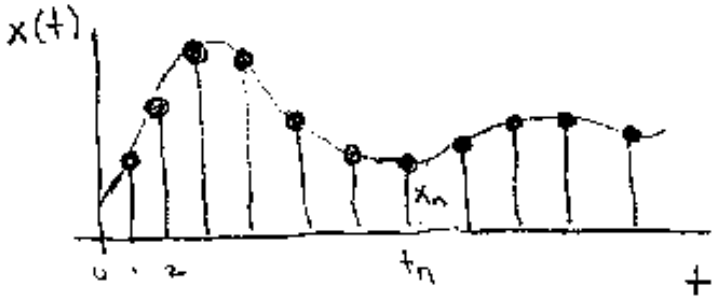
If
$$y_1(t) = \mathcal{H}\, x_1(t) \qquad (1)$$
$$y_2(t) = \mathcal{H}\, x_2(t)$$

Then
$$y_1(t) + y_2(t) = \mathcal{H}\left(x_1(t) + x_2(t) \right) \qquad (2)$$

Impulse Response (Green's Functions)



$x(t) \Rightarrow$ any function.



Sample $x(t)$ with a sampling function $\Delta(t)$ which forms an orthonormal set.

$$x(t) \approx \sum_n x(t_n)\Delta(t - t_n) \qquad (3)$$

where $\Delta(t) = 1$ for $t = 0$
$\quad\quad\quad\;\; = 0$ for $t \neq 0$

$$y(t) \approx \mathcal{H} \sum_n x(t_n)\Delta(t-t_n)$$

$$y(t) \approx \sum_n x(t_n)\,\mathcal{H}\Delta(t-t_n)$$

To get the most accurate representation of $y(t)$ place the sampling functions as close as possible.

$$\sum_n \Rightarrow \int d\tau \qquad \Delta(t) \Rightarrow \delta(t)$$

$$t_n \Rightarrow \tau$$

$$y(t) = \int_{-\infty}^{t} x(\tau)\left[\mathcal{H}\,\delta(t-\tau)\right] d\tau$$

Define the impulse response $h(t)$

$$h(t) = \mathcal{H}\,\delta(t)$$

$$y(t) = \int_{-\infty}^{t} x(\tau)\,h(t-\tau)\,d\tau$$

Which means we only have to solve the circuit <u>once</u> for $h(t)$ and can determine $y(t)$ for any $x(t)$.

$$y(t) = x(t) \otimes h(t)$$

Convolution is a compact way of writing down the circuit response but in general convolution integrals are messy to do.

The Laplace Transform

Since linear electrical circuits respond naturally in the "s" plane, why not try to represent the forcing function of the circuit in terms of a sum over all possible s plane solutions.

$$i_{gen}(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} I_{gen}(s)\, e^{st}\, ds$$

This is the inverse Laplace transform

$$i_{gen}(t) = \mathcal{L}^{-1}(I_{gen}(s))$$

Then

$$I_{gen}(s) = \int_0^{\infty} e^{-st}\, i_{gen}(t)\, dt$$

This is the Laplace transform.

$$I_{gen}(s) = \mathcal{L}(i_{gen}(t))$$

$$\mathcal{L}(\delta(t)) = 1$$

$$\mathcal{L}\left(\frac{d}{dt}\right) = \text{mult by } s \qquad I(s) = sC\,V(s)$$

$$\mathcal{L}\left(\int dt\right) = \text{division by } s \qquad I(s) = \frac{1}{sL}\,V(s)$$

Forced Response in the s domain

$$X(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$$

$$\mathcal{L}\left\{ y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau \right\}$$

$$\Downarrow$$

$$Y(s) = H(s)\, X(s)$$

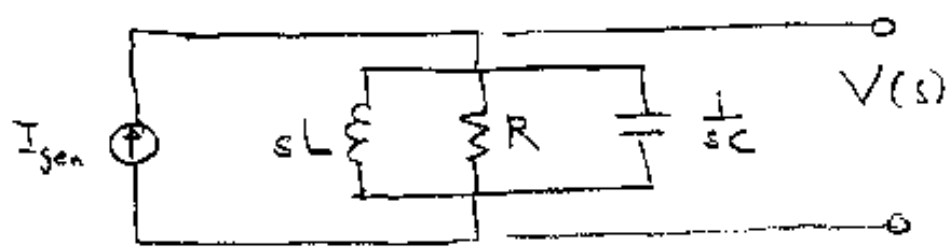
Convolution in time domain is <u>multiplication</u> in the s Domain!!

The impulse response:

$$H(s) = \frac{Y(s)}{X(s)} \leftarrow \text{Transfer function}$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\}$$

Return back RLC circuit.



Transfer function

$$V(s) = Z(s)\, I_{gen}(s)$$

Transfer function is actually an Impedance. in this case

$$Z(s) = \frac{R\left(\frac{s}{RC}\right)}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

Definitions: The <u>zero's</u> of a response are the values of s in which the numerator goes to zero.

The <u>pole's</u> of a response are the values of s in which the denominator goes to zero.

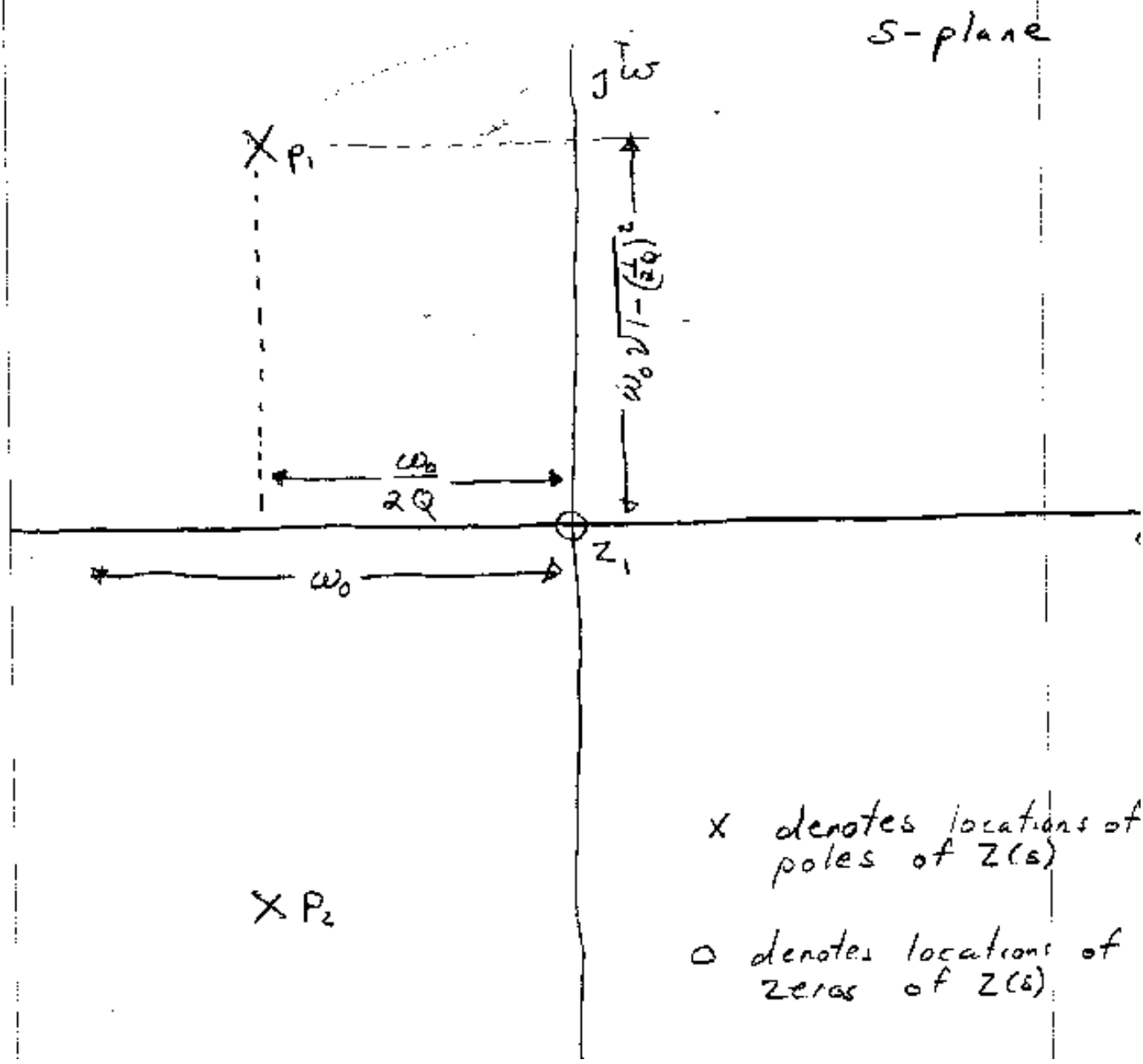
⊗ The poles of a response are also the roots of the Characteristic Eqn. This means that the transient or natural solution is determined by the poles of the response.

For a circuit to be stable we want the transient response to be bounded as time $\rightarrow \infty$. This means that for a stable circuit, the poles of the circuit must lie on the LEFT HAND SIDE of the s plane.

Define (as before) $\omega_0 = \frac{1}{\sqrt{LC}}$ $\quad Q = \omega_0 RC$

$$Z(s) = \frac{R\left(\frac{\omega_0}{Q}\right)s}{s^2 + \left(\frac{\omega_0}{Q}\right)s + \omega_0^2}$$

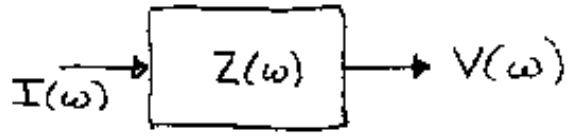
What good is knowing $Z(s)$? In an accelerator we don't use complex frequencies. We're interested in $Z(\omega)$.



$Z(\omega)$ is a complex number.

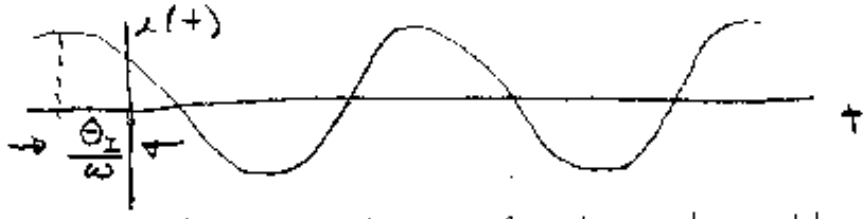
$$Z(\omega) = |Z| e^{j\theta_z}$$

What does this phase angle θ_z mean?



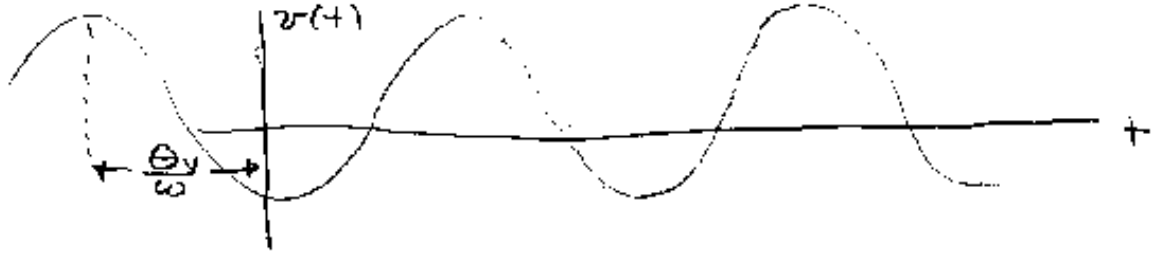
$I(\omega) = |I(\omega)| e^{j\theta_I}$ can be thought of as shorthand notation of a sinusoidal current in time

$$i(t) = |I| \cos(\omega t + \theta_I)$$



This current is then fed into the transfer function and the output voltage is:

$$v(t) = |V(\omega)| \cos(\omega t + \theta_V)$$



Since $V(\omega) = Z(\omega) I(\omega)$

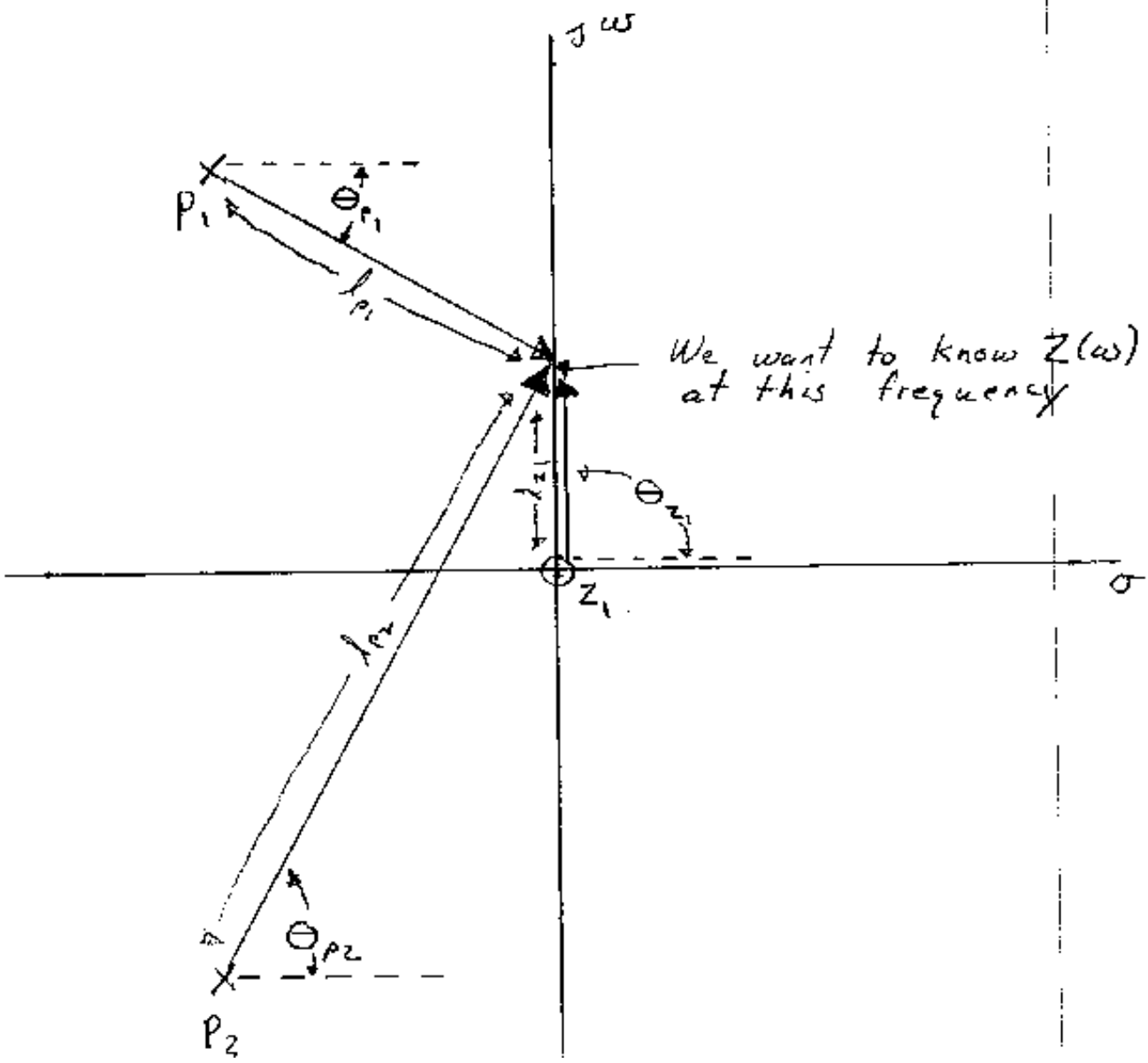
$$|Z(\omega)| = \frac{|V(\omega)|}{|I(\omega)|}$$

$$\theta_z = \theta_v - \theta_I$$

In general we can write our transfer function as

$$Z(s) = \frac{K\,(s - s_{z1})(s - s_{z2})\cdots(s - s_{zN})}{(s - s_{p1})(s - s_{p2})\cdots(s - s_{pM})}$$

Each term in a bracket can be thought of as a complex vector whose origin is located at the given pole or zero and is terminated on the $j\omega$ axis at the desired frequency



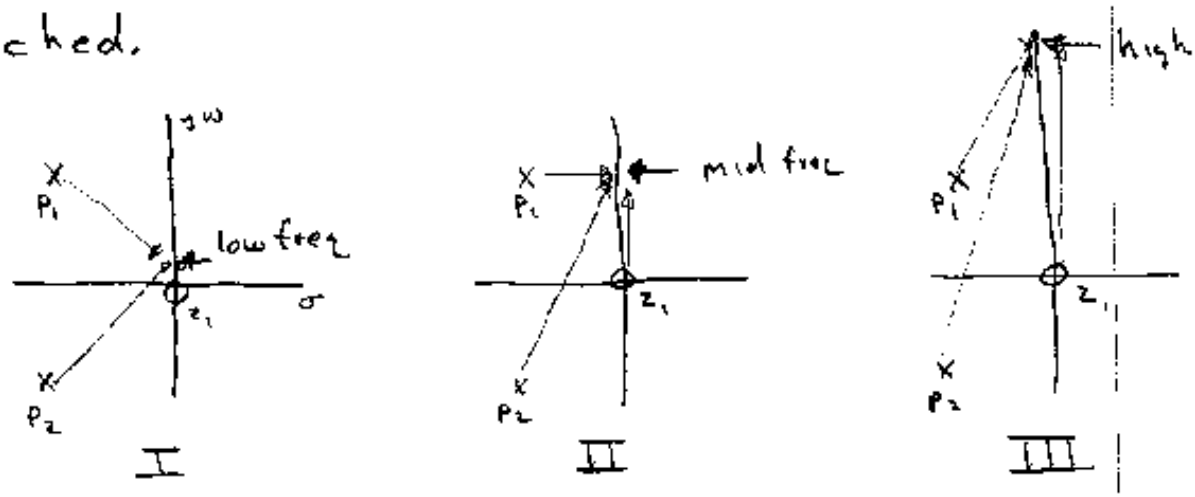
The magnitude of the transfer function $|Z(\omega)|$ is the product of the length of all of the zeros divided by the product of the length of all of the poles.

$$|Z(\omega)| = \frac{\prod_{n=1}^{N} l_{z_n}}{\prod_{m=1}^{M} l_{p_m}}$$

The phase of the transfer function is the sum of the angles of all of the zeros minus the sum of the angles of all of the poles.

$$\Theta_z = \sum_{n=1}^{N} \Theta_{z_n} - \sum_{m=1}^{M} \Theta_{p_m}$$

Using these simple rules the frequency response of a transfer function can be quickly sketched.



I. For low frequencies.

Magnitude: The length of the zero vector is very small so the <u>magnitude is small</u>.

Phase : The angle of the two poles are about opposite so they cancel. The angle of the zero is 90°. So the total <u>phase angle is 90°</u>.

II For mid frequencies:

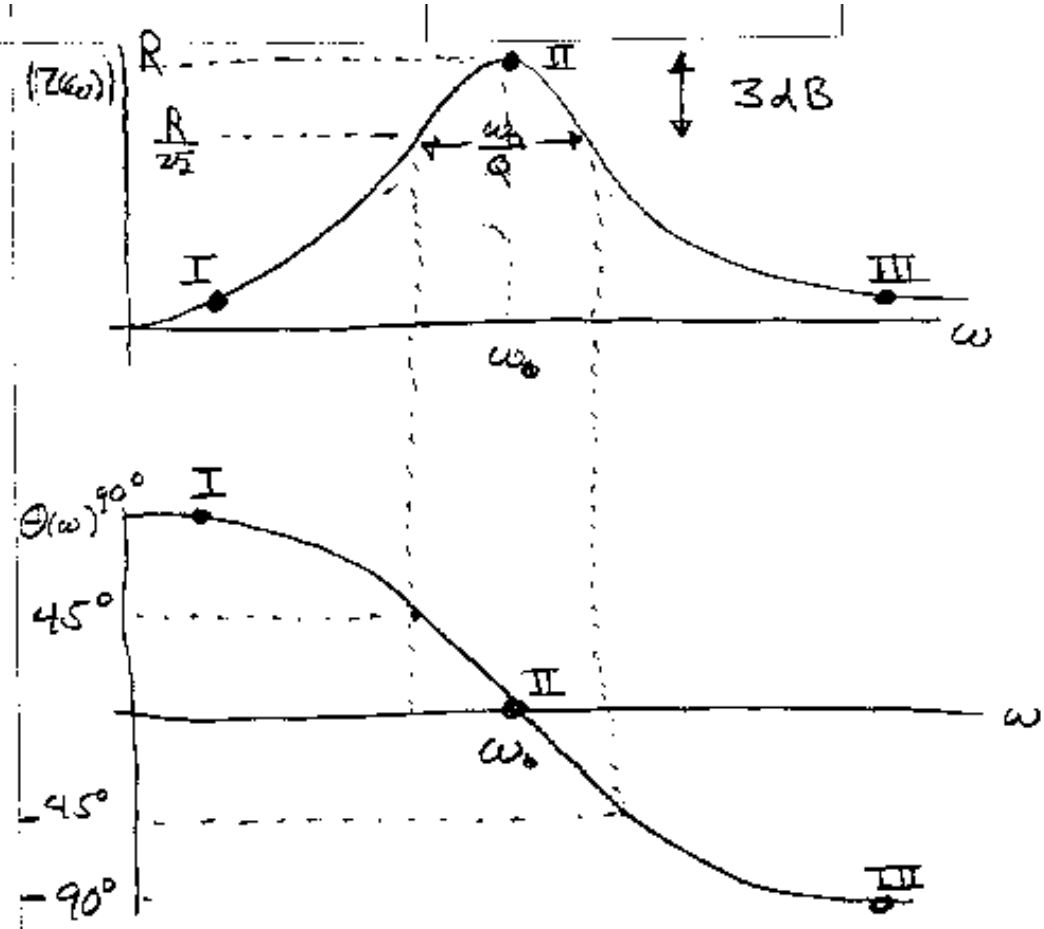
Magnitude: The length of the pole p_1 is very small. Since it is in the denominator the <u>magnitude is large</u>

Phase : The angle of the zero and the second pole are equal so they about cancel. The angle of the first pole is close to zero so the <u>total angle is close to zero</u>

III For high frequencies:

Magnitude: The length of the zero about cancels the length of one of the poles. However the length of the other pole is large so the <u>magnitude is small.</u>

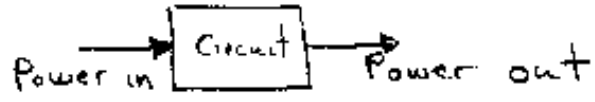
Phase : The angle of the zero and one of the poles about cancel but the angle of the other pole is 90°. <u>So the phase is −90°</u>



This is the transfer response of a resonant cavity. The peak is at ω_0. The 3dB bandwidth is $\frac{\omega_0}{Q}$.

What is a dB?

A dB is just a ratio of Power Gain



The power gain in dB's $= 10 \log_{10}\left(\frac{\text{Power out}}{\text{Power in}}\right)$

Since $P = \frac{V^2}{2R}$

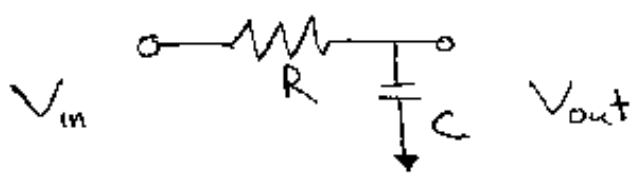
$$\text{Power gain in dB's} = 10 \log_{10}\left(\frac{\frac{V_{out}^2}{2R_{out}}}{\frac{V_{in}^2}{2R_{in}}}\right)$$

If $R_{out} = R_{in}$ (which is usually the case for amplifiers)

$$\text{Power gain in dB's} = 20 \log_{10}\left(\frac{V_{out}}{V_{in}}\right)$$

Because the power gain of amplifier chains are usually very large, dB's is a convenient way to write down the gain. Also for an amplifier chain one just adds the dB's to get the total gain.

<u>Another example</u>



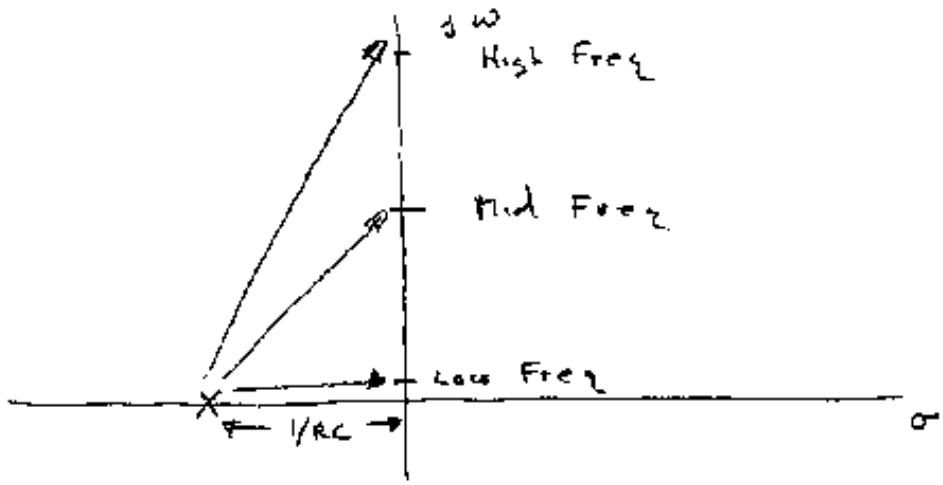
What is the transfer function $\frac{V_{out}}{V_{in}}$?

The circuit is just a voltage divider

$$V_{out} = \frac{\frac{1}{sC}}{\frac{1}{sC} + R} V_{in}$$

$$\frac{V_{out}}{V_{in}} = \frac{1}{1+sRC} = \frac{1}{RC}\left(\frac{1}{s+1/RC}\right)$$

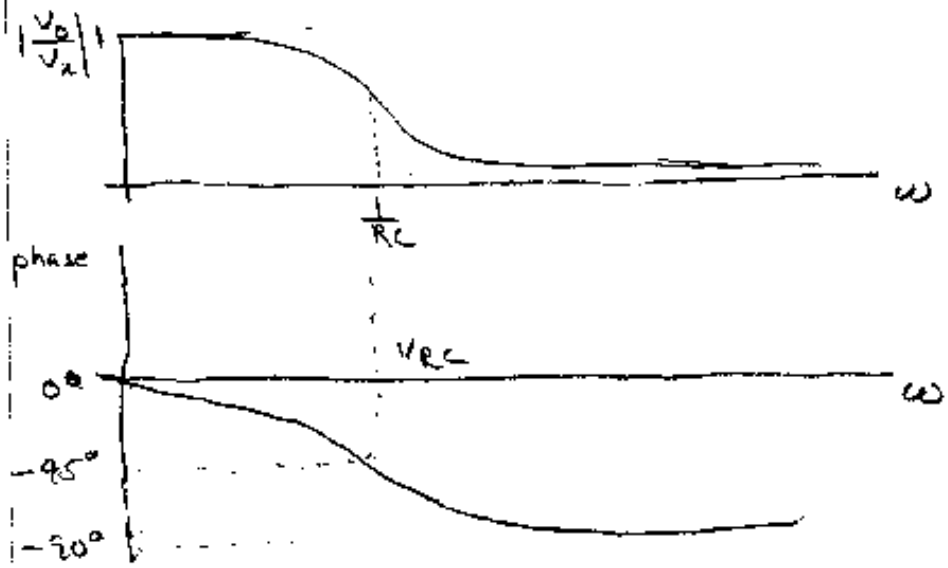
This has one pole at $s = -\frac{1}{RC}$



At Low freq, the length of the vector is $\frac{1}{RC}$ and the angle is $0°$

At Mid freq, the length of the vector is $\sqrt{2}\,\frac{1}{RC}$ so the response is down 3dB from the low freq response. The angle of the vector is $45°$

At High freq, the vector is very long and the angle is close to $90°$



SUBJECT: Examples

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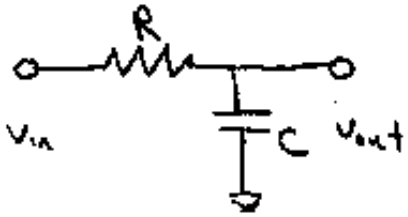
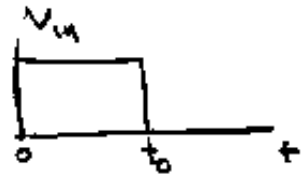




$$i_c = C\frac{dV_{out}}{dt} \qquad V_{in} = i_c R + V_{out}$$

$$V_{in} = RC\frac{dV_{out}}{dt} + V_{out}$$

i) Find impulse response

$$RC\frac{dh_{out}}{dt} + h_{out} = \delta(t-\tau)$$

Impulse $\delta(t-\tau) = 0 \qquad t \neq \tau$

$\Rightarrow \infty \qquad t = \tau$

$$\int_{-\infty}^{\infty} \delta(t-\tau)\,dt = 1$$

$$RC\frac{dh_{out}}{dt} + h_{out} = 0$$

$$h_{out} = A e^{-t/RC}$$

for $t < \tau \qquad V_{out} = 0$

$t > \tau \qquad V_{out} = A e^{-(t-\tau)/RC}$

$$\int_{\tau-\varepsilon}^{\tau+\varepsilon}\left(RC\frac{dh_{out}}{dt} + h_{out}\right)dt = \int_{\tau-\varepsilon}^{\tau+\varepsilon}\delta(t-\tau)\,dt$$

$$RC\, h_{out}\Big|_{\tau-\varepsilon}^{\tau+\varepsilon} = 1$$

$$RC\,A = 1$$

$$A = \frac{1}{RC}$$

$$h_{out}(t-\tau) = \frac{1}{RC}\, e^{-(t-\tau)/RC}$$

$$V_{out} = \int_{-\infty}^{t} V_{in}(\tau)\, h_{out}(t-\tau)\, d\tau$$

For $t < 0$ $\quad V_{out} = 0$

$0 < t < t_0$

$$V_{out} = \int_0^t V_{in} \frac{1}{RC} e^{-(t-\tau)/RC} d\tau$$

$$= V_{in}\, e^{-(t-\tau)/RC}\Big|_0^t$$

$$= V_{in}\left(1 - e^{-t/RC}\right)$$

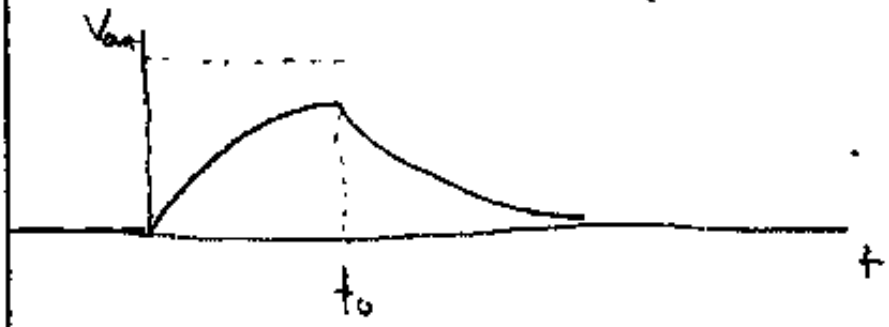
For $t > t_0$

$$V_{out} = \int_0^{t_0} V_{in} \frac{1}{RC} e^{-(t-\tau)/RC} d\tau$$

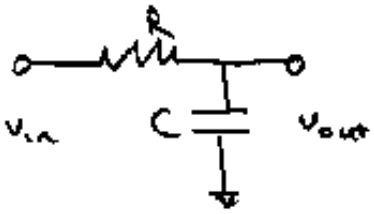
$$= V_{in}\, e^{-(t-\tau)/RC}\Big|_{0}^{t_0}$$

$$= V_{in}\, e^{-(t-t_0)/RC} - e^{-t/RC}$$

$$= V_{in}\, e^{-(t-t_0)/RC}\left(1 - e^{-\tau_0/RC}\right)$$



Laplace Domain.



$$\frac{V_{out}}{V_{in}} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}}$$

$$= \frac{1}{RC}\,\frac{1}{s + \frac{1}{RC}}$$

$$\mathcal{L}\{\delta(t-\tau)\} = e^{-s\tau}$$

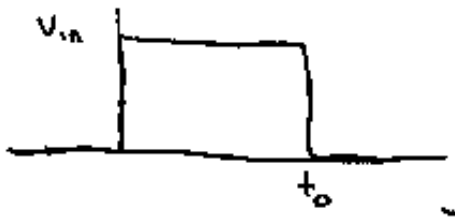
$$H_{out}(s) = \frac{e^{-s\tau}}{RC}\,\frac{1}{s + 1/RC}$$

$$h_{out}(t) = \frac{1}{RC}\, e^{-(t-\tau)/RC}\, u(t-\tau)$$

TABLE 2-3-1. LAPLACE TRANSFORM PAIRS

$f(t)$	$f(s)$
1. Unit impulse: $I(t)$	1. 1
2. Unit step: $u(t)$	2. $1/s$
3. Ramp: t	3. $1/s^2$
4. $t^n/n!$	4. $\frac{1}{s^{n+1}}$
5. e^{-at}	5. $\frac{1}{s+a}$
6. $\frac{t^n e^{-at}}{n!}$	6. $\frac{1}{(s+a)^{n+1}}$
7. $\sin \omega t$	7. $\frac{\omega}{s^2+\omega^2}$
8. $\cos \omega t$	8. $\frac{s}{s^2+\omega^2}$
9. $e^{-at} \sin \omega t$	9. $\frac{\omega}{(s+a)^2+\omega^2}$
10. $e^{-at} \cos \omega t$	10. $\frac{(s+a)}{(s+a)^2+\omega^2}$
11. $\frac{K}{\omega} e^{-\alpha t} \sin(\omega t + \psi)$	11. $\frac{(s+a_0)}{(s+\alpha)^2+\omega^2}$
$\psi = \tan^{-1} \frac{\omega}{a_0 - \alpha}$	
$K = [(a_0 - \alpha)^2 + \omega^2]^{1/2}$	
12. $\frac{d}{dt}[f(t)]$	12. $sf(s) - f(0)$
13. $\frac{d^2}{dt^2}[f(t)]$	13. $s^2 f(s) - sf(0) - \frac{d}{dt} f(0)$
14. $\int_0^t f(t)\, dt$	14. $\frac{f(s)}{s} + \frac{1}{s} \int f(t)\, dt \mid_0$
15. $f(t-a)u(t-a)$	15. $e^{-as} f(s)$

equation shows that differentiation in the time domain is equiva
plying by s in the Laplace domain. The Laplace transform of an i



$$V_{in} = V_{in}\left(u(t) - u(t - t_0)\right)$$

$$\mathcal{L}(V_{in}) = V_{in}\left(\frac{1}{s} - \frac{e^{-st_0}}{s}\right)$$

$$V_{out} = \frac{V_{in}}{RC}\,\frac{1}{s + 1/RC}\left(\frac{1}{s} - \frac{e^{-st_0}}{s}\right)$$

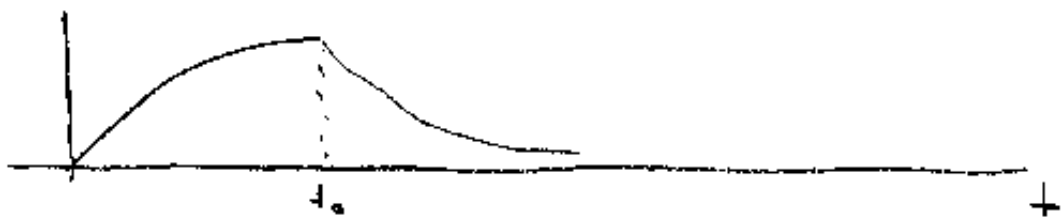
$$\left(\frac{1}{s}\right)\frac{1}{s + 1/RC} = \frac{A}{s} + \frac{B}{s + 1/RC}$$

$$= \frac{A(s + 1/RC) + Bs}{s(s + 1/RC)}$$

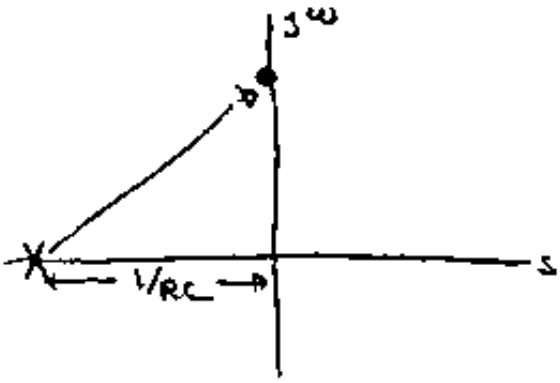
$$A + B = 0 \qquad \frac{A}{RC} = 1 \qquad A = RC$$

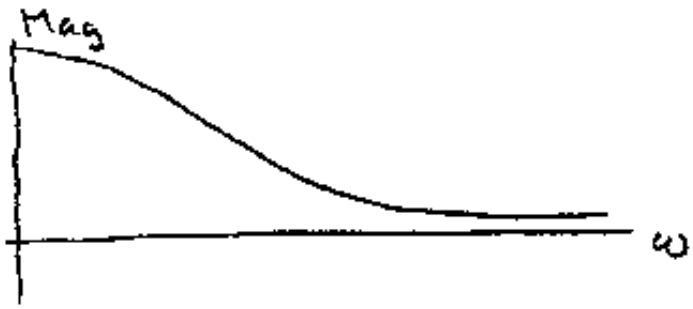
$$V_{out} = V_{in}\left(\frac{1}{s} - \frac{1}{s + 1/RC}\right) - V_{in}e^{-st_0}\left(\frac{1}{s} - \frac{1}{s + 1/RC}\right)$$

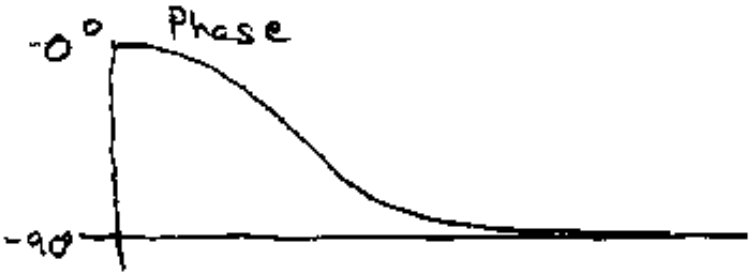
$$= V_{in}\left(u(t) - e^{-t/RC}\right) - V_{in}\left(u(t - t_0) - e^{-(t - t_0)/RC}\right)$$

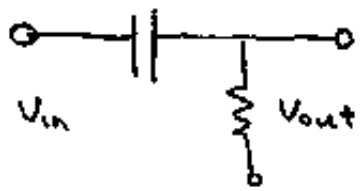


Pole - zero plot









$$i_c = C\frac{d(V_{in} - V_{out})}{dt} = \frac{V_{out}}{R}$$

$$RC\frac{dV_{out}}{dt} + V_{out} = RC\frac{dV_{in}}{dt}$$

$$RC\frac{d\,h_{out}}{dt} + h_{out} = RC\frac{d}{dt}\left(\delta(t)\right)$$

Derivative of a Delta Function $\Rightarrow$ YUK!

Let's do this in the Laplace Domain

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{R}{R + \frac{1}{sC}} = \frac{s}{s + \frac{1}{RC}} = H(s)$$

$$H(s) = A + \frac{B}{s + 1/RC} = \frac{As + A\frac{1}{RC} + B}{s + 1/RC}$$

$A = 1 \qquad B = -1/RC$

$$H(s) = 1 - \frac{1}{RC}\frac{1}{s+1/RC}$$

$$h(t) = \delta(t) - \frac{1}{RC}e^{-t/RC}$$

$$V_{in}(t) = V_{in}\left(u(t) - u(t-t_0)\right)$$

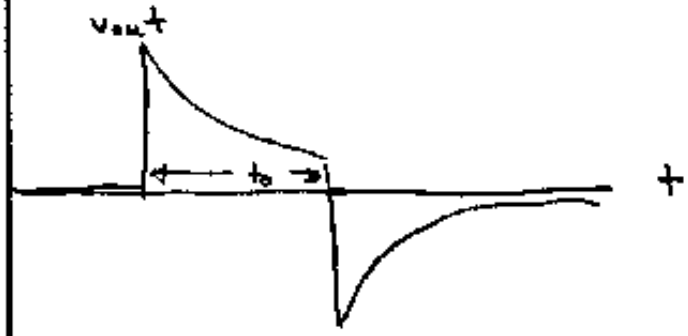
$$\mathcal{L}(V_{in}) = \frac{V_{in}}{s} - \frac{e^{-st_0}}{s} = V_{in}(s)$$

$$H(s)\,V_{in}(s) = V_{in}\left(1 - \frac{1}{RC}\frac{1}{s+1/RC}\right)\left(\frac{1}{s} - \frac{e^{-st_0}}{s}\right)$$

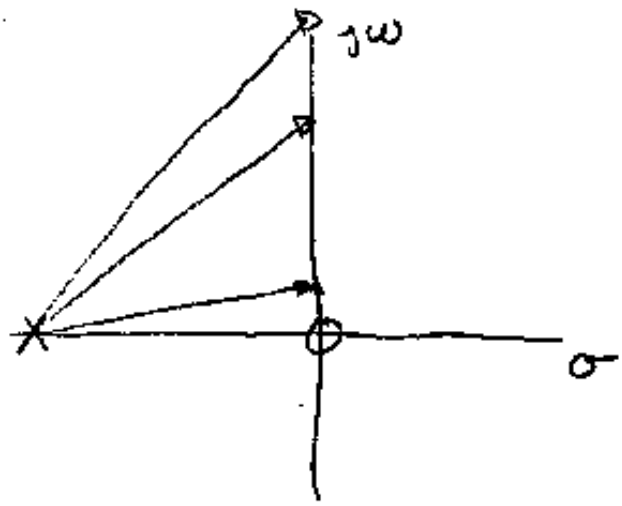
$$\frac{1}{s} - \frac{1}{RC}\left(\frac{1}{s}\right)\left(\frac{1}{s+\frac{1}{RC}}\right) = \frac{1}{s} - \frac{1}{RC}\left(\frac{RC}{s} - \frac{RC}{(s+1/RC)}\right)$$

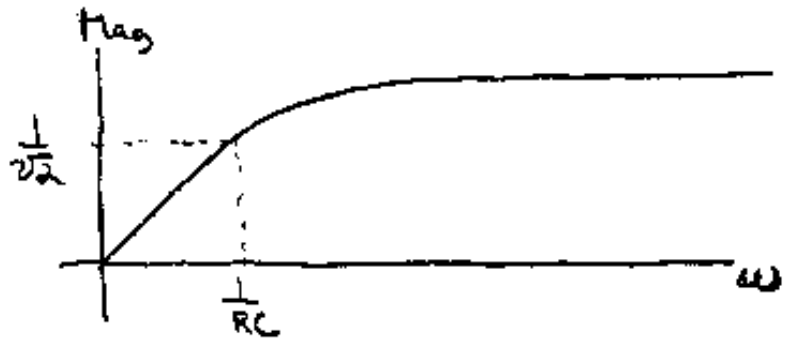
$$= \frac{1}{s+1/RC}$$

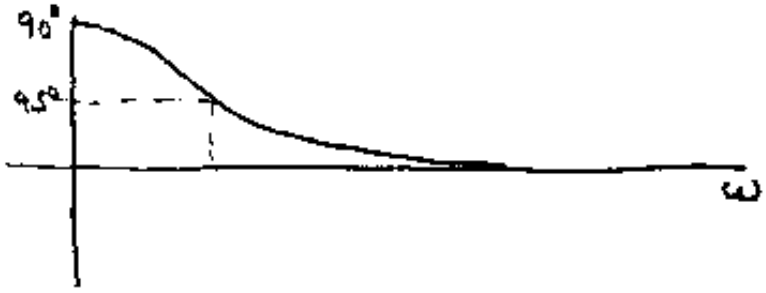
$$V_{out}(t) = u(t)e^{-t/RC} - u(t-t_0)e^{-(t-t_0)/RC}$$



Pole - Zero Plot







Power Spectral Density

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} v(t)\, i(t)\, dt$$

$$i(t) = \frac{v(t)}{R}$$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{v(t)\, v(t)}{R}\, dt$$

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(\omega)\, e^{j\omega t}\, d\omega$$

$$P = \frac{1}{2\pi} \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{V(\omega_1)\, V(\omega_2)}{R} e^{j\omega_1 t}\, e^{j\omega_2 t} \cdot d\omega_1\, d\omega_2\, dt$$

For a real function

$$V(-\omega) = V^*(\omega)$$

$$\int_{-\infty}^{\infty} V(\omega_2)\, e^{j\omega_2 t}\, d\omega_2 = \int_{-\infty}^{\infty} V(\omega_2)^*\, e^{-j\omega_2 t}\, d\omega_2$$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2\, \frac{V(\omega_1)\, V^*(\omega_2)}{R} \frac{1}{2\pi} \int_{-T/2}^{T/2} e^{j(\omega_1-\omega_2)t}\, dt$$

$$V(\omega_1) = \int_{-\infty}^{\infty} v(t)\, e^{-j\omega_1 t}\, dt$$

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(\omega_2)\, e^{j\omega_2 t}\, d\omega_2$$

$$V(\omega_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(\omega_2)\, d\omega_2 \int_{-\infty}^{\infty} e^{j(\omega_2-\omega_1)t}\, dt$$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j(\omega_2-\omega_1)t}\, dt = \delta(\omega_2-\omega_1)$$

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2\, \frac{V(\omega_1)V^*(\omega_2)}{R}\, \delta(\omega_1-\omega_2)$$

$$= \int_{-\infty}^{\infty} d\omega_1 \left[\lim_{T\to\infty} \frac{1}{T}\, \frac{|V(\omega_1)|^2}{R}\right]$$

$$= \int_{-\infty}^{\infty} df \left[\lim_{T\to\infty} \frac{2\pi}{T}\, \frac{|V(f_1)|^2}{R}\right]$$

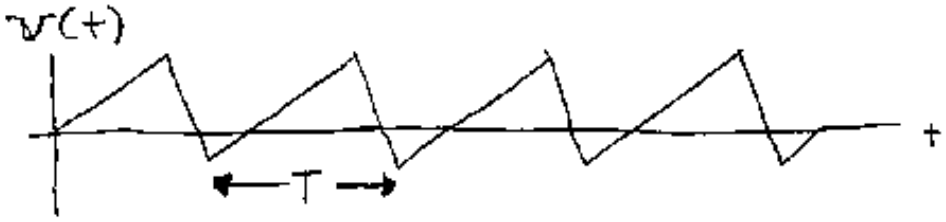
Spectral Density

$$\boxed{\begin{aligned} S(\omega) &= \lim_{T\to\infty} \frac{1}{T}\, \frac{|V(\omega)|^2}{R} \\ P &= \int_{-\infty}^{\infty} S(\omega)\, d\omega \end{aligned}}$$

Fourier Transforms

Laplace transforms can be used to calculate the transient & forced response. That is why the complex frequency has two variables ($\sigma, j\omega$). However, most of the time we are interested only in the forced response. For this we need only one frequency variable $j\omega$.

Consider a circuit which is excited by a periodic wave

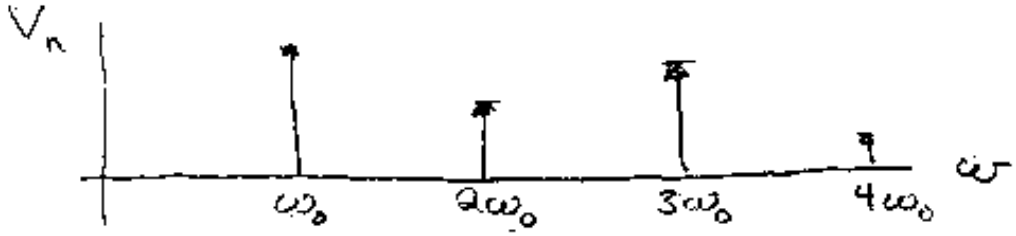


This wave can be represented as a Fourier series:

$$v(t) = \sum_{n=-\infty}^{\infty} V_n e^{jn\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T}$

The coefficients V_n form a <u>discrete</u> spectrum



The coefficients V_n are given as

$$V_n = \frac{1}{T}\int_{-T/2}^{T/2} v(t)\, e^{-jn\omega_0 t}\, dt$$

Note the V_n is a complex number that has a <u>magnitude</u> and a <u>phase</u>.

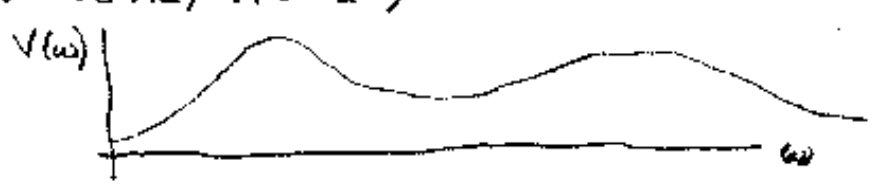
Now go to the limit of $v(t)$ not having a well defined period. At this point a discrete spectrum will not do. So:

$$v(t) = \int_{-\infty}^{\infty} V(\omega)\, e^{j\omega t}\, d\omega$$

where the summation has been replaced with an integral. The period T in which the coefficient $V(\omega)$ is calculated over now must stretch to infinity

$$V(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} v(t)\, e^{-j\omega t}\, dt$$

The coefficient $V(\omega)$ maybe thought of as a spectral <u>density</u> (Note that the units of $V(\omega)$ is Volts/Hertz)



Two Golden Rules of Discrete Fourier Transforms

The integral to calculate the spectral density can be approx. as:

$$V(\omega) \approx \frac{1}{2\pi} \sum_{n=-N}^{N} v(n\Delta t)\, e^{jn\omega\Delta t}\, \Delta t$$

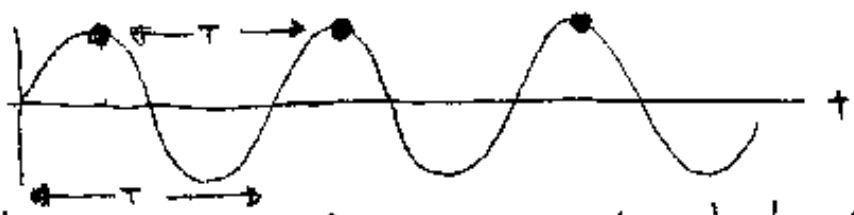
which is the Discrete Fourier Transform. (DFT)

There are 2 parameters that describe the DFT:

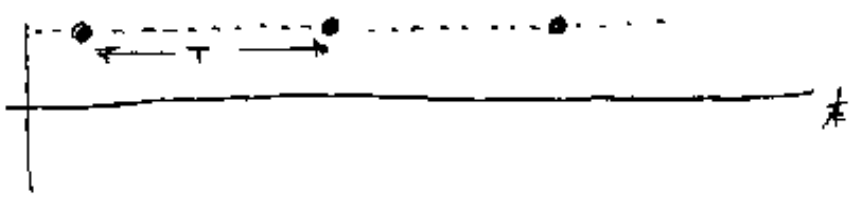
$\frac{1}{\Delta t}$ = the sample rate

$N\Delta t$ = the sample length.

Nyquist sampling: Consider a sine wave with a period of T. Assume that our sampling rate is also T.

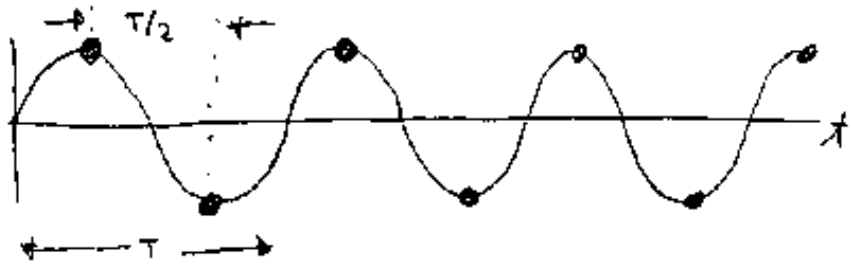


What would the waveform reconstructed from samples look like:

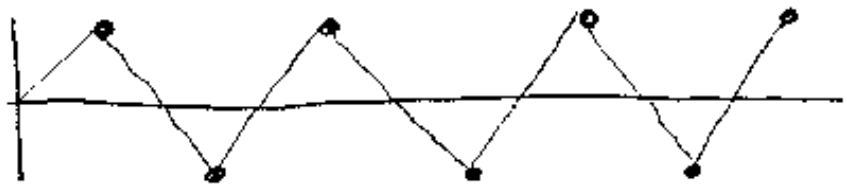


which looks like a DC voltage.

Now assume that the Sample rate is twice the RF frequency



The reconstructed wave form

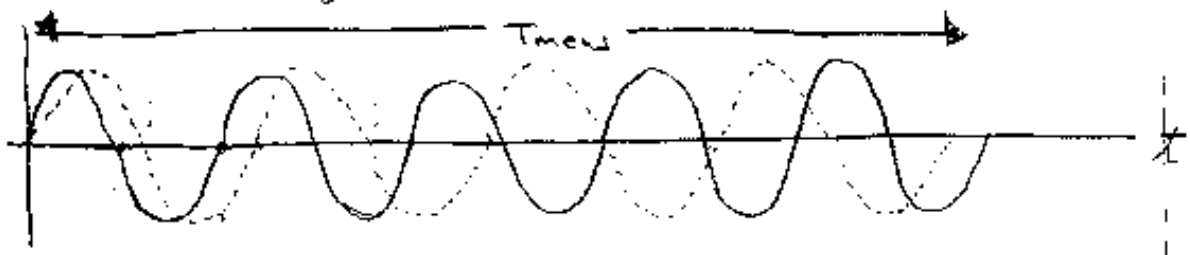


which is a much better approx. to the wave form.

> ① The sampling rate must be at least <u>twice</u> the frequency of the highest frequency component in the waveform

<u>Resolution</u>: What is the uncertainty in a frequency measurement?

We can measure frequencies by counting the number of periods in a measurement interval of length T.



— f_1 ---- f_2

Number of periods for f_1 is $T_{meas} f_1$

Number of periods for f_2 is $T_{meas} f_2$

We can tell the difference between the two when

$$1 < T_{meas} f_1 - T_{meas} f_2 \qquad \Delta f = f_1 - f_2$$

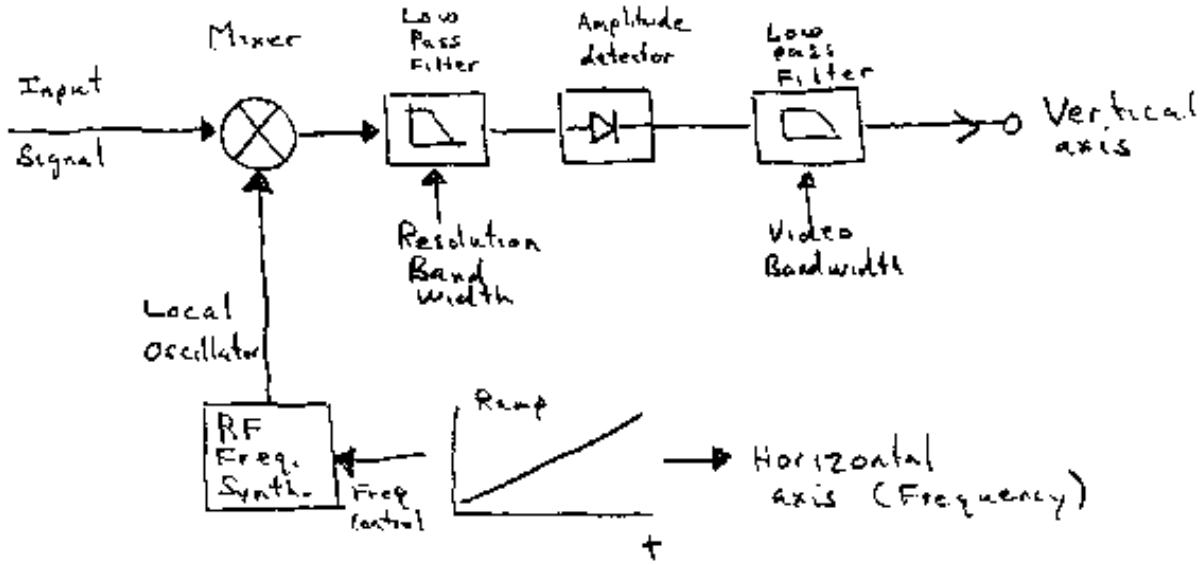
$$\Delta f > \frac{1}{T_{meas}}$$

② The resolution in frequency is inversely proportional to the length of the measurement. This is just a re-statement of the uncertainty principle

$$\Delta f \, \Delta t \geq 2\pi$$

Spectrum Analyzers

How do spectrum analyzers work?



A mixer is just a multiplier

Input signal $= V_{in} \cos(\omega_{in} t)$

Local Oscillator $= \cos(\omega_{osc} t)$

The output of the mixer is.

$$V_{mixer} = V_{in}\cos(\omega_{in}t)\cos(\omega_{osc}t)$$

Trig identity $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos(\alpha)\cos(\beta)$$

$$V_{mixer} = \frac{V_{in}}{2}\cos((\omega_{in}+\omega_{osc})t) + \frac{V_{in}}{2}\cos((\omega_{in}-\omega_{osc})t)$$

This term is high frequency & will not make it thru the LPF

This term will make it thru the LPF if $|\omega_{in}-\omega_{osc}| < \omega_{LPF_{cutoff}}$

The first filter determines the resolution of the analyzer. It's cutoff freq. is called the resolution bandwidth. Remember that the cutoff freq. of the filter is inversely proportional to the size of the capacitor in the filter. But, the bigger the capacitor, the longer it takes to charge it up. So for very narrow filters, the meas. will take a long time to complete.

After the Resolution bandwidth filter

$$V_{LPF} = \frac{V_{in}}{2}\cos((\omega_{in}-\omega_{osc})t)$$

The next stage detects the amplitude

$$V_{Amp\,der} \propto V_{in}.$$

What if V_{in} varies very slowly with time.

$$V_{in} \Rightarrow V_{in}(t)$$

The video bandwidth filter will average these variations.

The output of the spectrum analyzer is usually in dBm. A dBm is a <u>power</u> <u>level</u> defined as

$$P(dBm) = 10 \log_{10}\left(\frac{P_{out}}{1mW}\right)$$

So $-10 dBm = .1 mW$

$0 dBm = 1 mW$

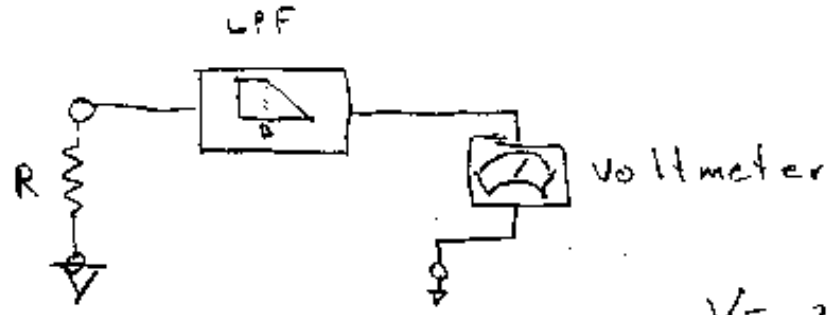
$10 dBm = 10 mW$ etc.

The spectrum analyzer actually measures how much power has passed thru the resolution bandwidth filter. So the actual units of measurement is in Watts/Hertz. This is the power spectral density of the signal.

Power detected = Spectral Density * Bandwidth

For a single tone the density is infinitely big but the bandwidth is infinitely small so the Power is finite

White Noise



$$V = \sqrt{kTBR}$$

k = Boltzman constant

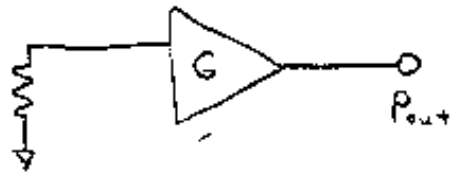
T = Absolute temperature

B = bandwidth of lowpass filter

$$P = \frac{V^2}{R} = kTB$$

Spectral Power Density $= kT$

Noise Figure



Noise Figure: $F = \dfrac{\text{Noise added by amp} + \text{Noise of input res}}{\text{Noise of input res.}}$

$$F = \frac{P_N + kTB}{kTB} \qquad F_{dB} = 10 \log_{10}(F)$$

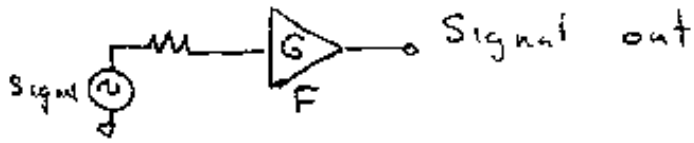
$F > 1$ or $F_{dB} > 0\,dB$

Effective noise temperature

$$P_N = (F-1)kTB$$
$$P_N = kT_{eff}B$$

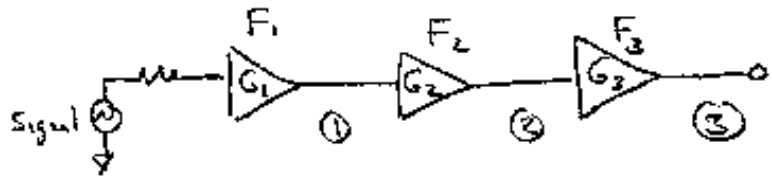
$$T_{eff} = (F-1)T$$

Amplifier chains



Signal out = G signal in
Noise out = $GFkTB$

$$\frac{\text{Signal}}{\text{Noise}} = \frac{\text{Signal}}{FkTB}$$



Noise at ① $= G_1F_1kTB$
at ② $= G_1G_2F_1kTB + G_2(F_2-1)kTB$
at ③ $= G_1G_2G_3F_1kTB + G_2G_3(F_2-1)kTB + G_3(F_3-1)k$

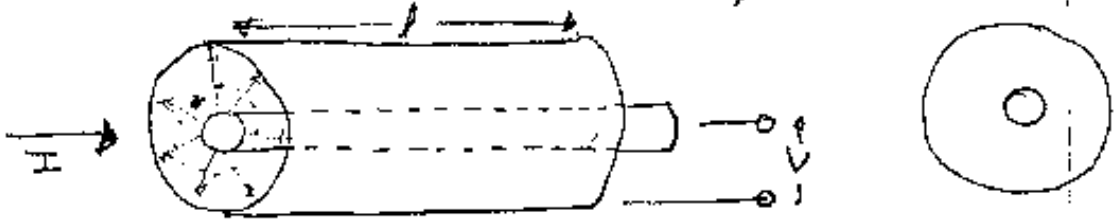
Signal at ③ $= G_1G_2G_3$ Signal

$$\frac{\text{Noise}}{\text{Signal}} \text{ at ③} = F_1kTB + \frac{(F_2-1)}{G_1}kTB + \frac{(F_3-1)}{G_1G_2}kTB$$

$$= F_{total}kTB$$

$$F_{total} = F_1 + \frac{F_2-1}{G_1} + \frac{F_3-1}{G_1G_2}$$

TEM Transmission Line Theory



Charge on inner conductor:

$$q = C l V$$

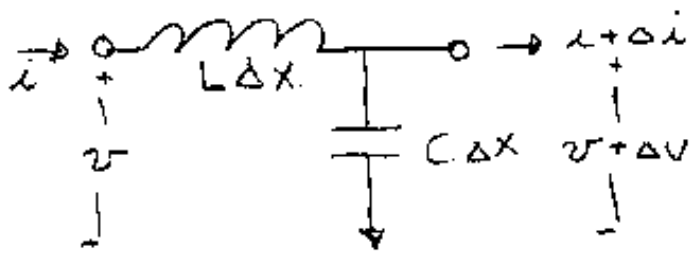
where C is the capacitance per unit length

Azimuthal magnetic flux:

$$\Phi = L l I$$

where L is the inductance per unit length.

Electrical model of a small section of coax.



Voltage drop along the inductor

$$v - (v + \Delta v) = L \Delta x \frac{\partial i}{\partial t}$$

Current flowing thru the capacitor

$$i + \Delta i = i - C \Delta x \frac{\partial v}{\partial t}$$

Limit as $\Delta \to 0$

$$-\frac{\partial v}{\partial x} = L \frac{\partial i}{\partial t} \qquad -\frac{\partial i}{\partial x} = C \frac{\partial v}{\partial t}$$

Solutions are traveling waves

$$v(t,x) = v^{+}\left(t - \frac{x}{vel}\right) + v^{-}\left(t + \frac{x}{vel}\right)$$

+ indicates wave traveling in the + direction
- indicates wave traveling in the - direction

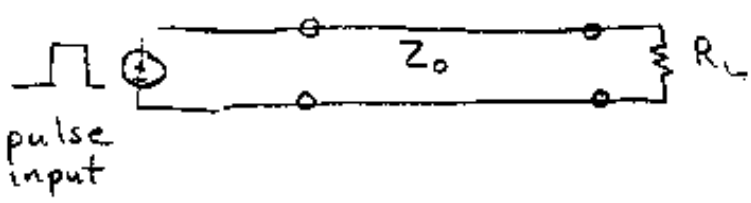
$$i = \frac{1}{Z_0} v^{+}\left(t - \frac{x}{vel}\right) - \frac{1}{Z_0} v^{-}\left(t + \frac{x}{vel}\right)$$

where vel is the phase velocity of the wave

$$vel = \frac{1}{\sqrt{LC}}$$

$$= \frac{1}{\sqrt{\mu\varepsilon}} \quad \text{for a TEM wave}$$

$$Z_0 = \sqrt{\frac{L}{C}}$$ has units of Ω's and is called the characteristic impedance



Pulse travels down the transmission line as a forward going current & voltage wave only. However, when pulse reaches load resistor we must have:

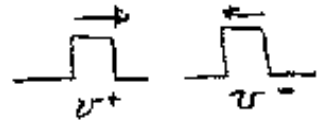
$$\frac{v}{i} = R_L = \frac{v^{+} + v^{-}}{\frac{v^{+}}{Z_0} - \frac{v^{-}}{Z_0}}$$

So a reverse wave v^- & i^- must be created to satisfy the boundary condition imposed by the load resistor.

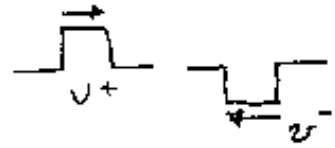
$$\frac{v^-}{v^+} = \frac{R_L - Z_0}{R_L + Z_0} = \Gamma \quad \text{(reflection coeff.)}$$

Three cases:

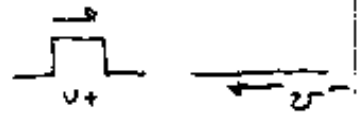
$R_L = \infty$ (open) $\quad \Gamma = +1$



$R_L = 0$ (short) $\quad \Gamma = -1$



$R_L = Z_0 \quad \Gamma = 0$



$\therefore$ A transmission line terminated with it's characteristic impedance looks the same as an infinitely long line.

<u>Sinusoidal Waves</u>

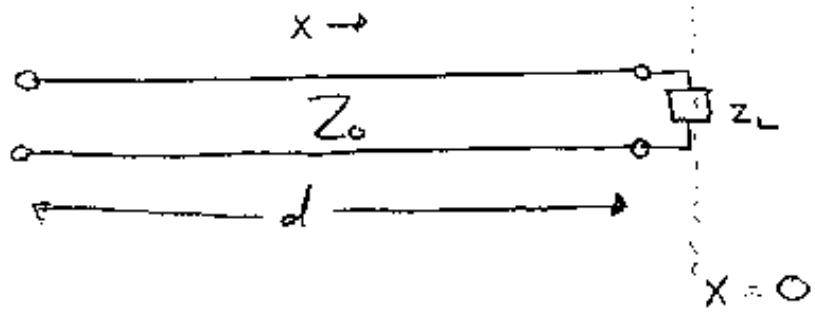
$$v^+ = V^+ \cos(\omega t - \beta x) = \mathrm{Re}\left\{ V^+ e^{-j\beta x} e^{j\omega t} \right\}$$

where $\frac{\omega}{\beta} = \text{vel} = \text{phase velocity}$

$$\beta = \frac{2\pi f}{\text{vel}} \quad \text{or} \quad \boxed{\beta = \frac{2\pi}{\lambda}}$$

By using sine waves we can now use complex impedances such as

$$Z_{capacitor} = \frac{1}{j\omega C} \qquad Z_{inductor} = j\omega L$$



at $x=0$ we have $V^- = \Gamma V^+ = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

$$V = V^+ e^{-j\beta x} + \Gamma V^+ e^{+j\beta x}$$

$$= V^+(1+\Gamma-\Gamma)e^{-j\beta x} + \Gamma V^+ e^{+j\beta x}$$

$$= V^+(1-\Gamma)e^{-j\beta x} + 2V^+\Gamma\left(\frac{e^{-j\beta x} + e^{+j\beta x}}{2}\right)$$

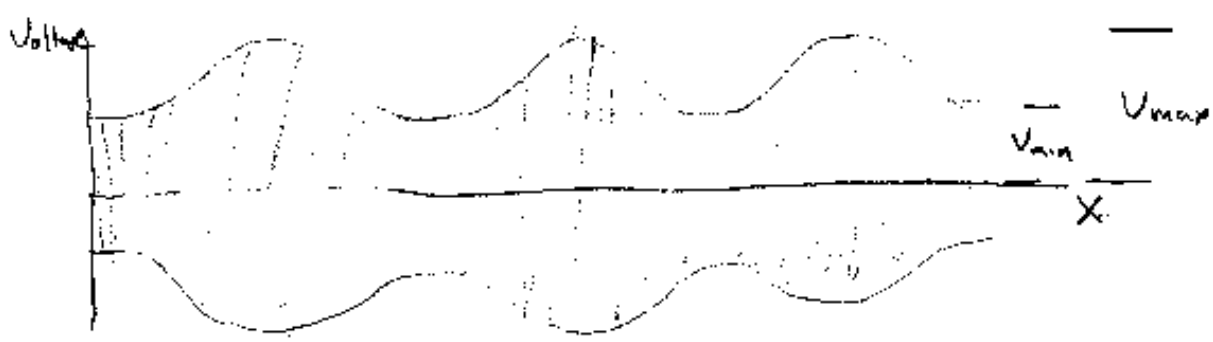
$$= \underbrace{V^+(1-\Gamma)e^{-j\beta x}}_{\text{travelling wave}} + \underbrace{2V^+\Gamma \cos\beta x}_{\text{standing wave}}$$

$$\frac{V_{max}}{V_{min}} = \frac{V^+(1+|\Gamma|)}{V^-(1-|\Gamma|)} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

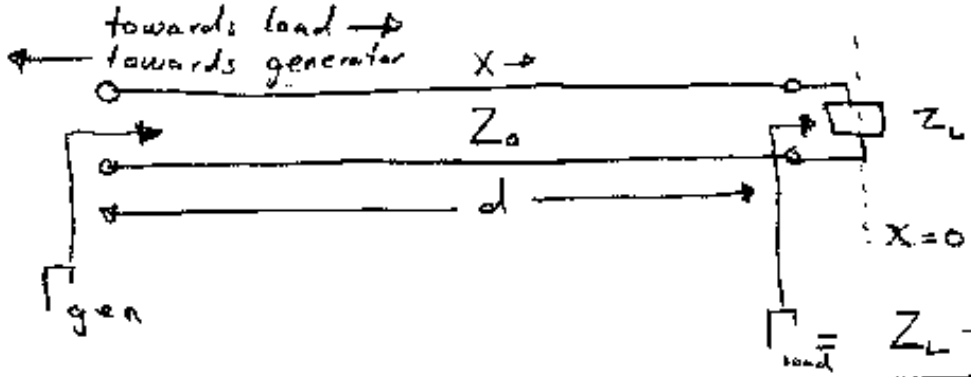
$$= \text{VSWR}$$

voltage standing wave ratio

$\text{VSWR} \geq 1$



Reflection Coefficient along a transmission line



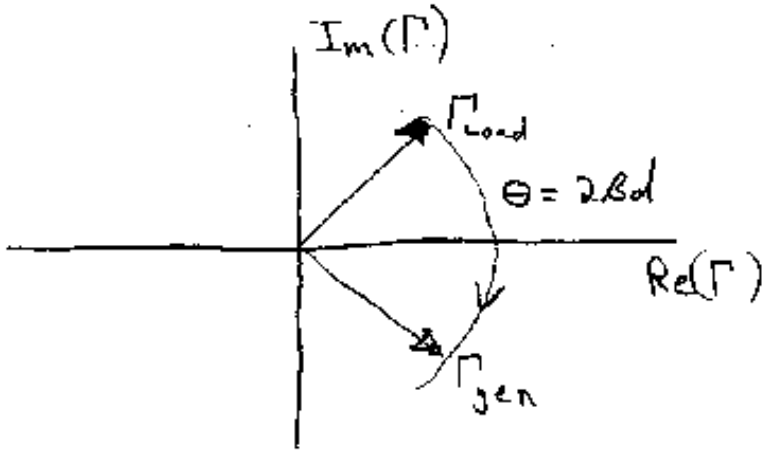
$$\Gamma_{load} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$V = V^+ e^{-j\beta x} + \Gamma_{load} V^+ e^{+j\beta x}$$

at $x = -d$ (at generator what is ratio of reverse wave to forward wave

$$\left.\frac{V_{forward}}{V_{reverse}}\right|_{gen} = \frac{\Gamma_{load} V^+ e^{-j\beta d}}{V^+ e^{j\beta d}} \qquad (z = -d)$$

$$\Gamma_{gen} = \Gamma_{load} e^{-j2\beta d}$$



There is still a one to one correspondence of Γ with load impedance

$$\Gamma = \frac{Z - Z_0}{Z + Z_0}$$

$$Z = Z_0\left(\frac{1+\Gamma}{1-\Gamma}\right)$$

$$Z = Z_0\left(\frac{1+\Gamma_L e^{-j2\beta d}}{1-\Gamma_L e^{-j2\beta d}}\right)$$

<u>For $\Gamma_L = 1$ (open)</u> $\quad Z = -jZ_0 \cot \beta d$

for $\beta d \ll 1$ $\quad Z \approx -j\frac{Z_0}{\beta d}$

$$= -j\frac{\sqrt{\frac{L}{C}}}{d\,\omega\sqrt{LC}}$$

$$= \frac{1}{j\omega C d} \quad \text{(capacitive)}$$

for $\beta d = \frac{\pi}{2}$ or $d = \frac{\lambda}{4}$

$$Z = 0$$

$\therefore$ an open looks like a short 1/4 wavelength away from the open.

<u>For $\Gamma_L = -1$ (short)</u> $\quad Z = jZ_0 \tan \beta d$

for $\beta d \ll 1$ $\quad Z \approx jZ_0 \beta d$

$$= j\sqrt{\frac{L}{C}}\, d\,\omega\sqrt{LC}$$

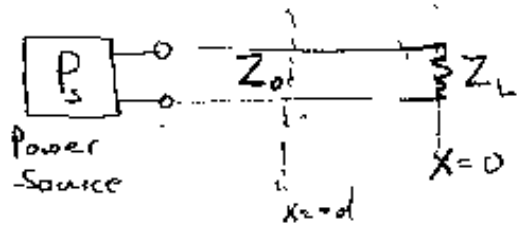
$$= j\omega L d \quad \text{(inductive)}$$

for $\beta d = \frac{\pi}{2}$ or $d = \frac{\lambda}{4}$

$$Z = \infty$$

∴ a short looks like an open 1/4 wavelength away from the short.

<u>Incident and reflected power.</u>



Voltage along the transmission line at $x=-d$

$$V(x=-d) = V_{inc}\, e^{j\beta d} + \Gamma V_{inc}\, e^{-j\beta d}$$

Current at $x=-d$

$$I(x=-d) = \frac{V_{inc}}{Z_0} e^{j\beta d} - \Gamma \frac{V_{inc}}{Z_0} e^{-j\beta d}$$

The rate of <u>Energy</u> flowing thru the plane at $x=-d$ is equal to

$$P = \frac{1}{2} \mathrm{Re}\left\{ V I^* \right\}$$

$$= \frac{1}{2} \frac{V_{inc}^2}{Z_0} \mathrm{Re}\left\{ 1 - |\Gamma|^2 + \Gamma e^{-j2\beta d} - \Gamma^* e^{j2\beta d} \right\}$$

form only an imaginary number

$$P = \frac{V_{inc}^2}{2 Z_0} \left(1 - |\Gamma|^2 \right)$$

$$P = \frac{V_{inc}^2}{2Z_0} - \frac{|\Gamma|^2 V_{inc}^2}{2Z_0}$$

incident power

reflected power

* Remember: Power does not flow, Energy does. The forward & reverse traveling waves are <u>power orthogonal</u> so that the net rate of energy transfer is equal to the difference in power of the individual waves.

To maximize the power absorbed by the load we want

$$\Gamma = 0$$

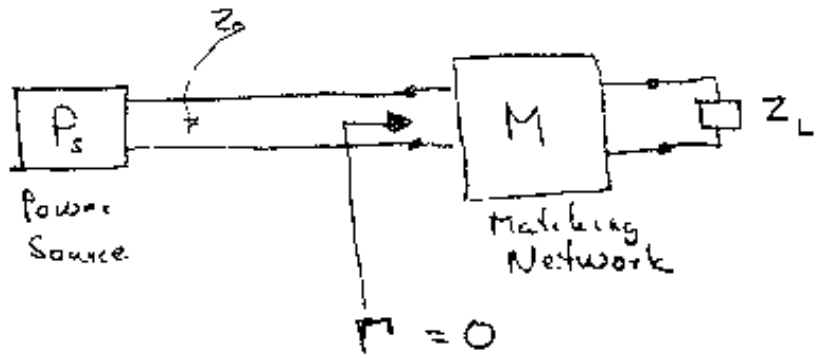
or

$$Z_L = Z_0$$

When $Z_L = Z_0$ we say that the load is matched to the transmission line.

<u>Smith Charts</u>

What if our load cannot be made equal to Z_0 for some other reasons? We need to build a matching network so that the source effectively sees a match load.



Normalized impedance

$$z = \frac{Z}{Z_0} = r + jx$$

The reflection coefficient is a complex number

$$\Gamma = u + jv$$

$$z = \frac{1 + \Gamma}{1 - \Gamma}$$

$$r = \frac{1 - u^2 - v^2}{(1-u)^2 + v^2}$$

$$x = \frac{2v}{(1-u)^2 + v^2}$$

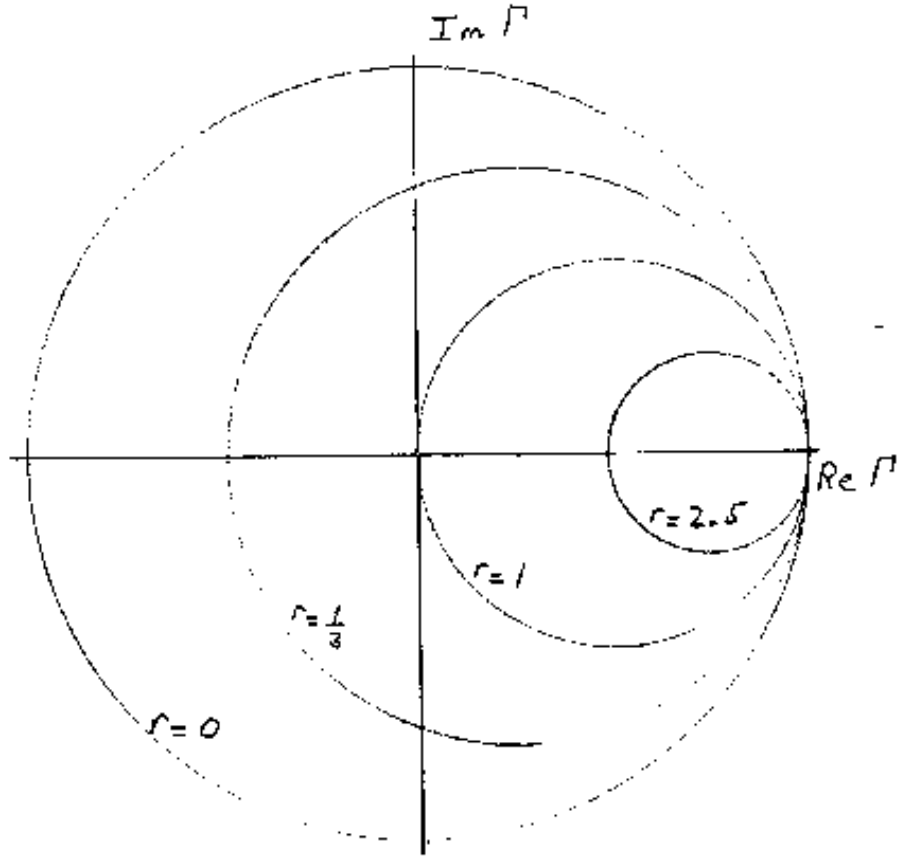
OR:

$$\left(u - \frac{r}{1+r}\right)^2 + v^2 = \frac{1}{(1+r)^2}$$

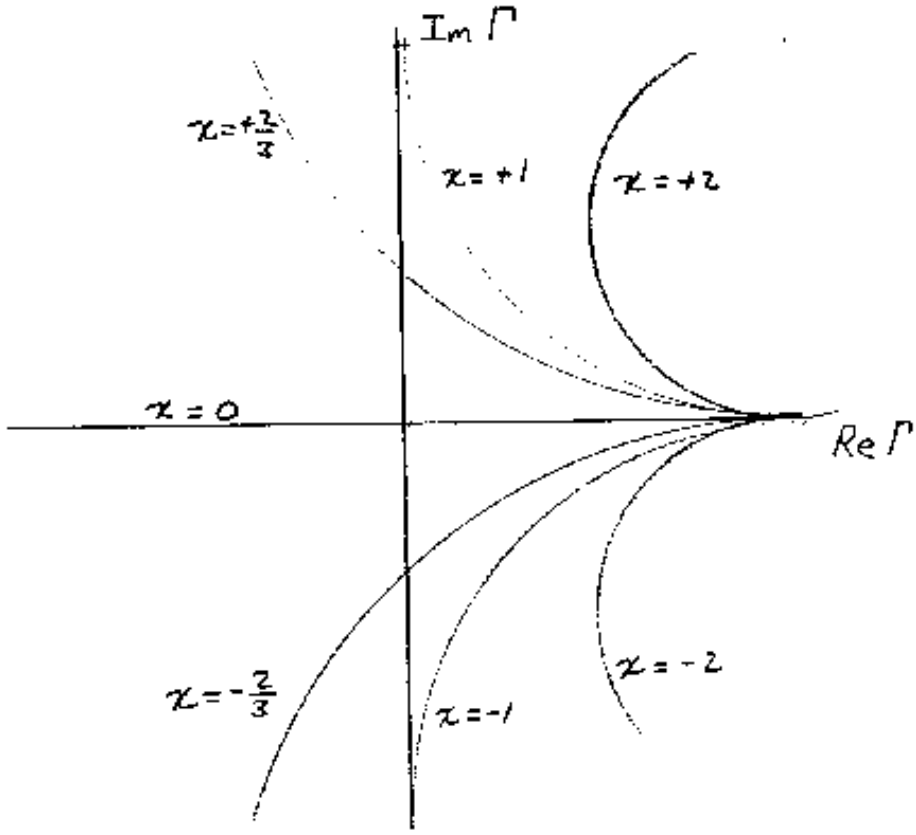
$$(u-1)^2 + \left(v - \frac{1}{x}\right)^2 = \frac{1}{x^2}$$

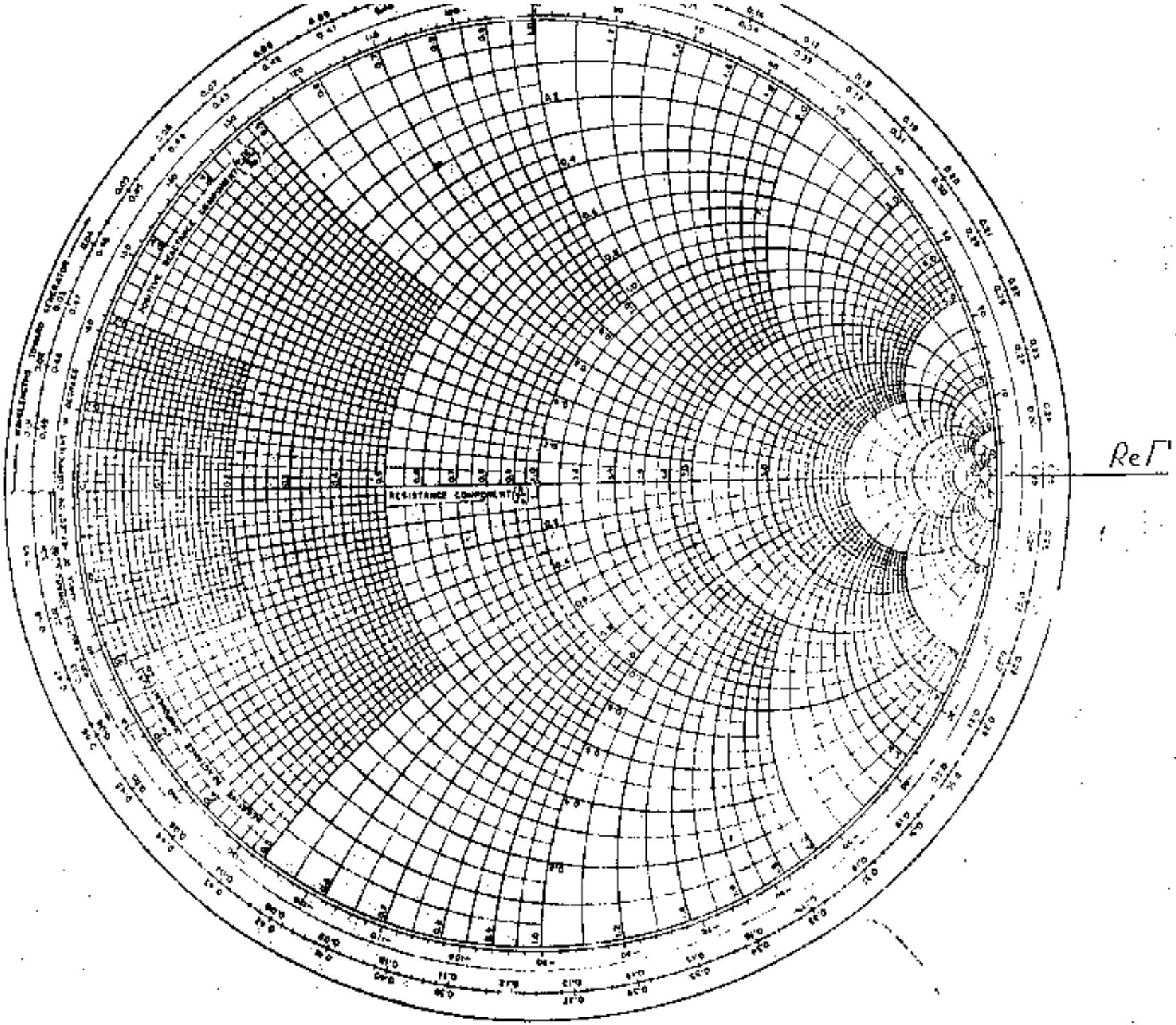
These are eqns. for circles on the u, v plane.

r circles



x circles




RESISTANCE COMPONENT
Re Γ

① Given $\Gamma_L = .5\angle 45^\circ$ $\quad Z_o = 50\Omega$

what is Z_L

$$Z_L = 50\Omega(1.35 + j1.35)$$
$$= 67.5\Omega + j67.5\Omega$$

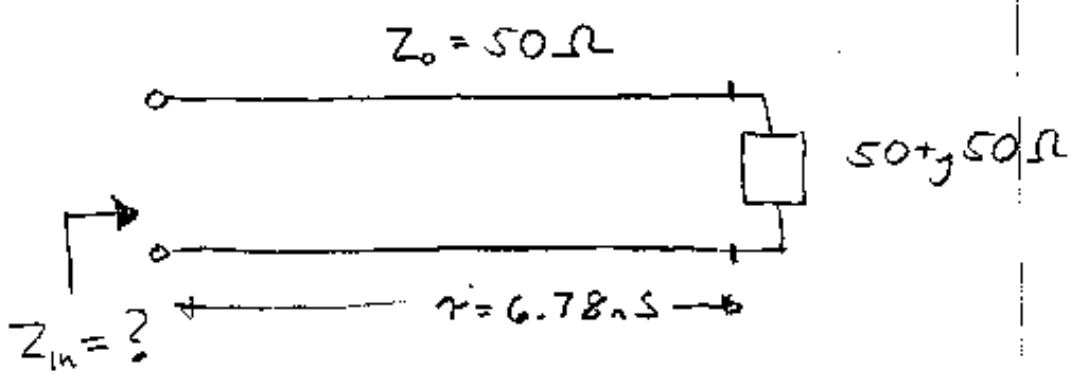
② Given $Z_L = 15\Omega - j25\Omega$ $\quad Z_o = 50\Omega$

what is Γ_L

$$z_L = .3 - j.5$$

$$\Gamma_L = .618\angle -124^\circ$$

③



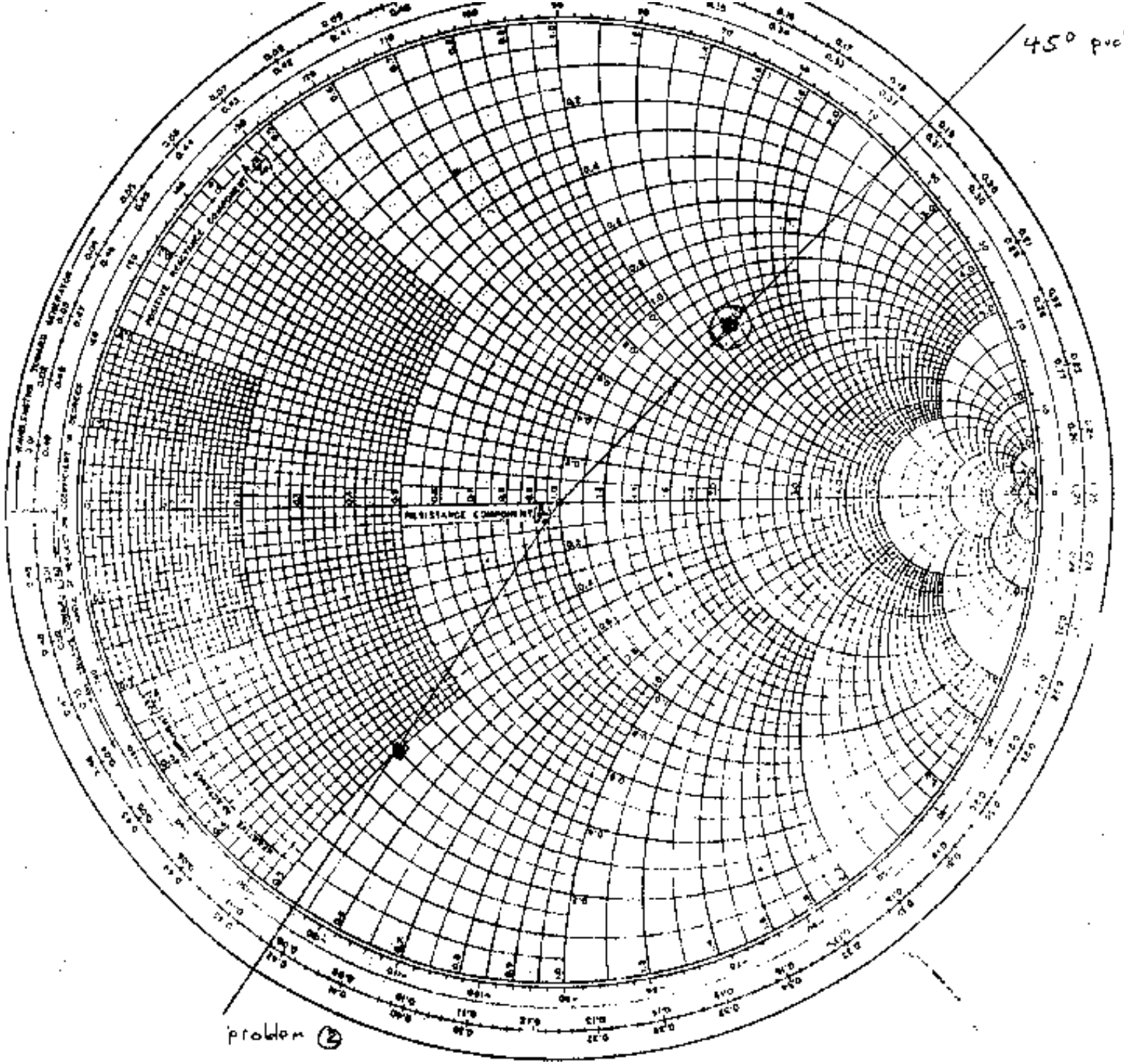
at 50MHz

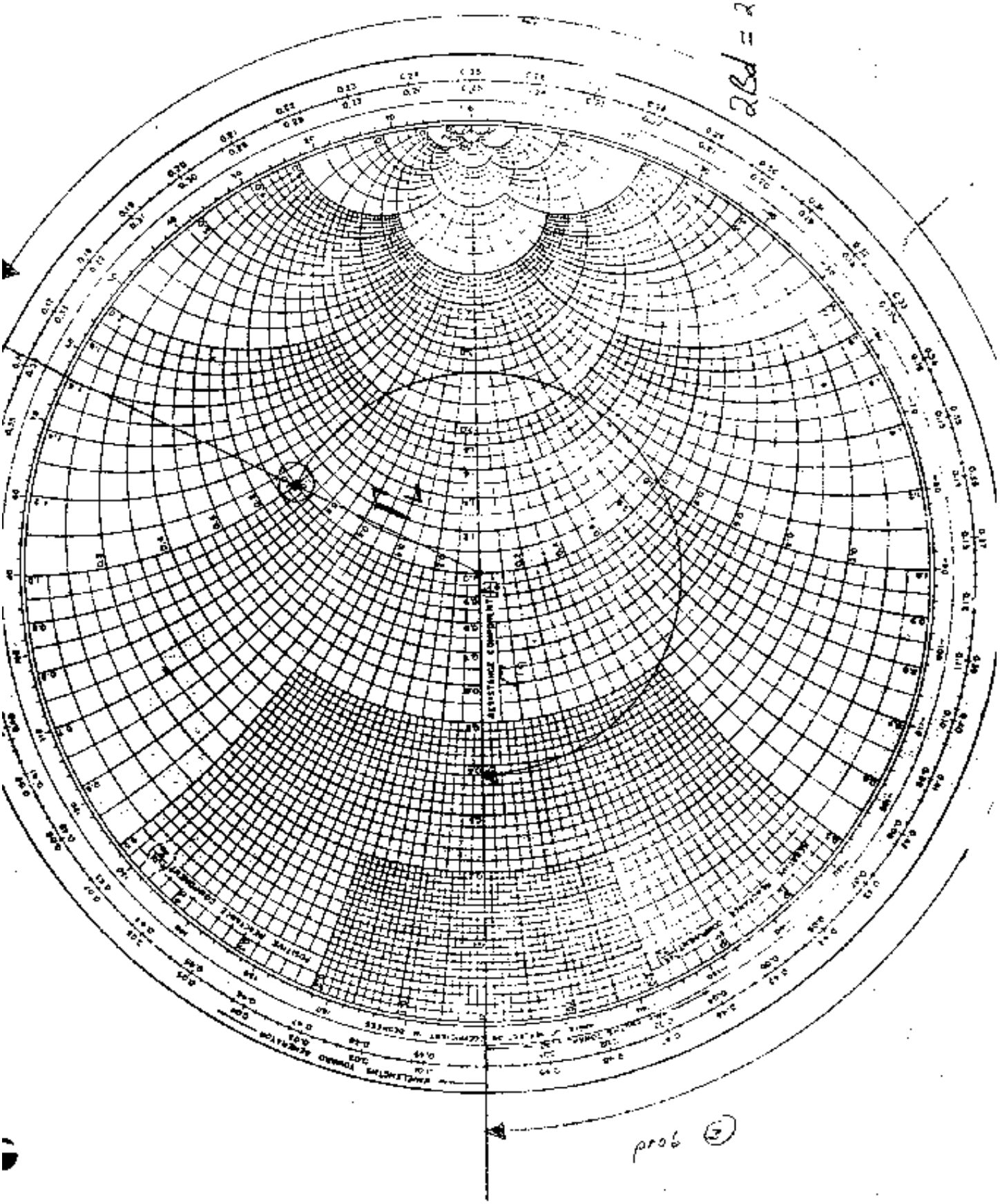
$$z_L = 1 + j1$$

$$\Gamma_L = .445\angle 64^\circ$$

$$\Gamma_{in} = \Gamma_L e^{-j2\beta d}$$

$$2\beta d = 2 \cdot \frac{2\pi f}{vel} \cdot d$$


45° problem ①
problem ②
RESISTANCE COMPONENT



$2Bd = 244°$

prob (3)

$$2\beta d = 2 \cdot 360^\circ \cdot f \cdot \tau \qquad \left(\frac{d}{vel} = \tau\right)$$

$$= 244^\circ$$

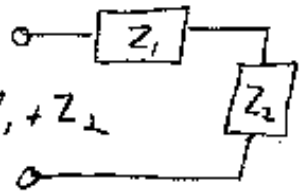
$$\Gamma_{in} = .445 \angle -180^\circ$$

$$Z_{in} = 50\Omega(0.38 + j\,0)$$

$$Z_{in} = 19\,\Omega$$

<u>Admittance</u>

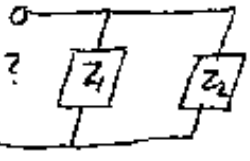
Impedance is nice to work with series configurations. For example:



$$V = ZI$$

$$Z_L = Z_1 + Z_2$$

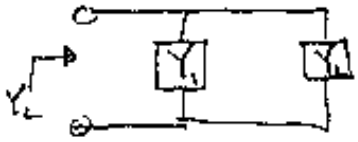
But not so nice for parallel configurations



$$Z_L = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

For parallel loads it is easier to work with admittance

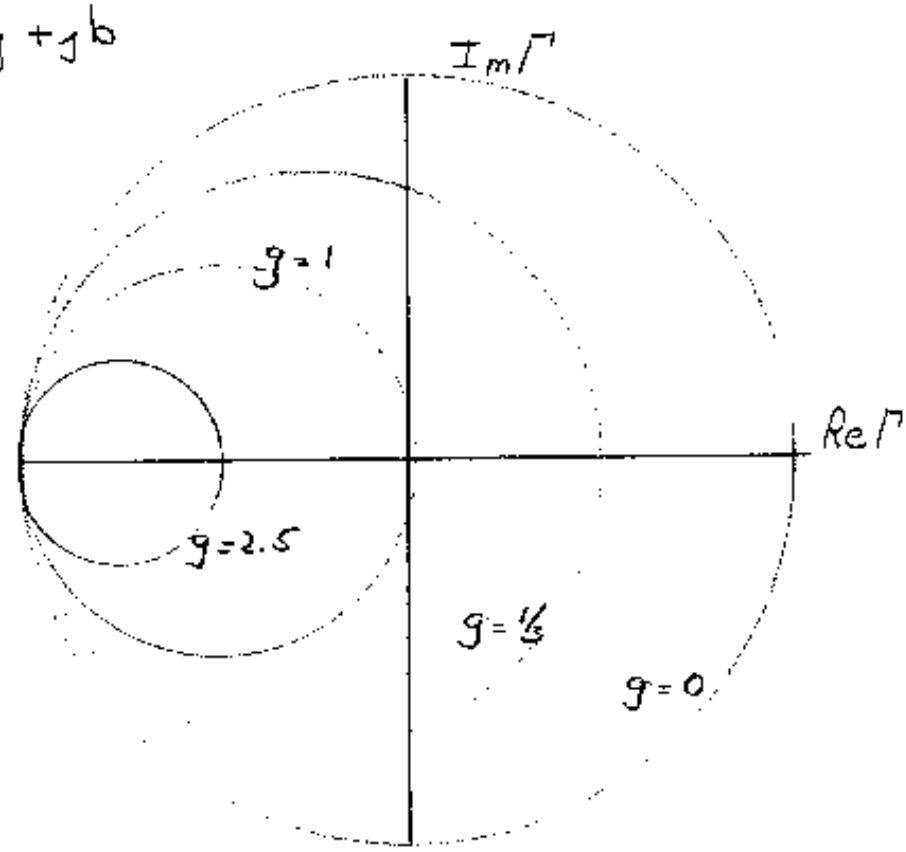
$$I = YV$$

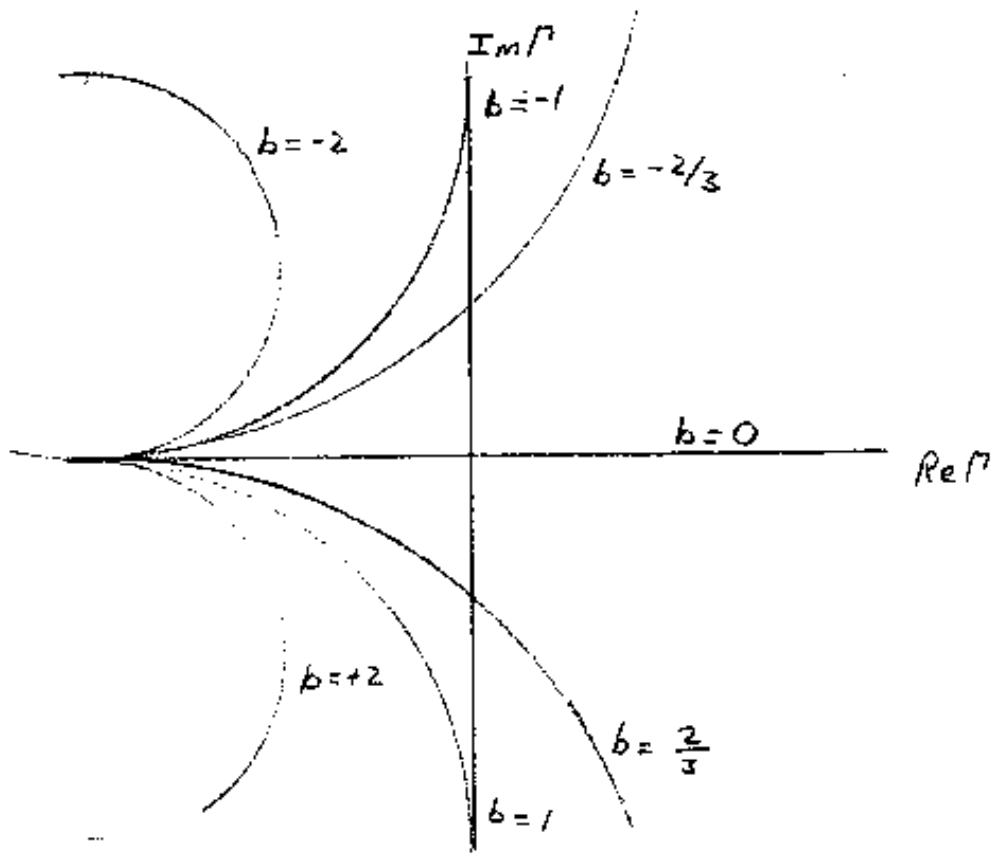


$$Y_1 = \frac{1}{Z_1} \qquad Y_2 = \frac{1}{Z_2} \qquad Y_L = Y_1 + Y_2$$

$$y = \frac{Y}{Y_0} = g + jb$$

$$Y_0 = \frac{1}{Z_0}$$

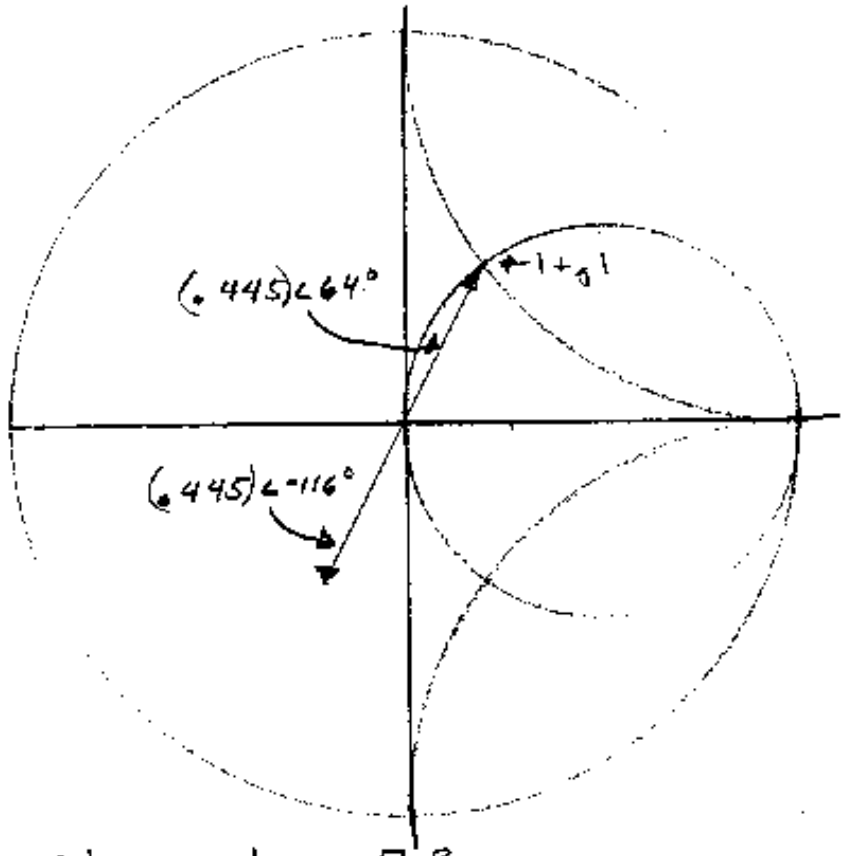




We see that the admittance Smith Chart is the impedance Smith Chart rotated 180°. We could use TWO charts one for impedance & one for admittance

OR:

We could use one chart & just flip the reflection coefficient vector 180°



<u>Example 1</u>

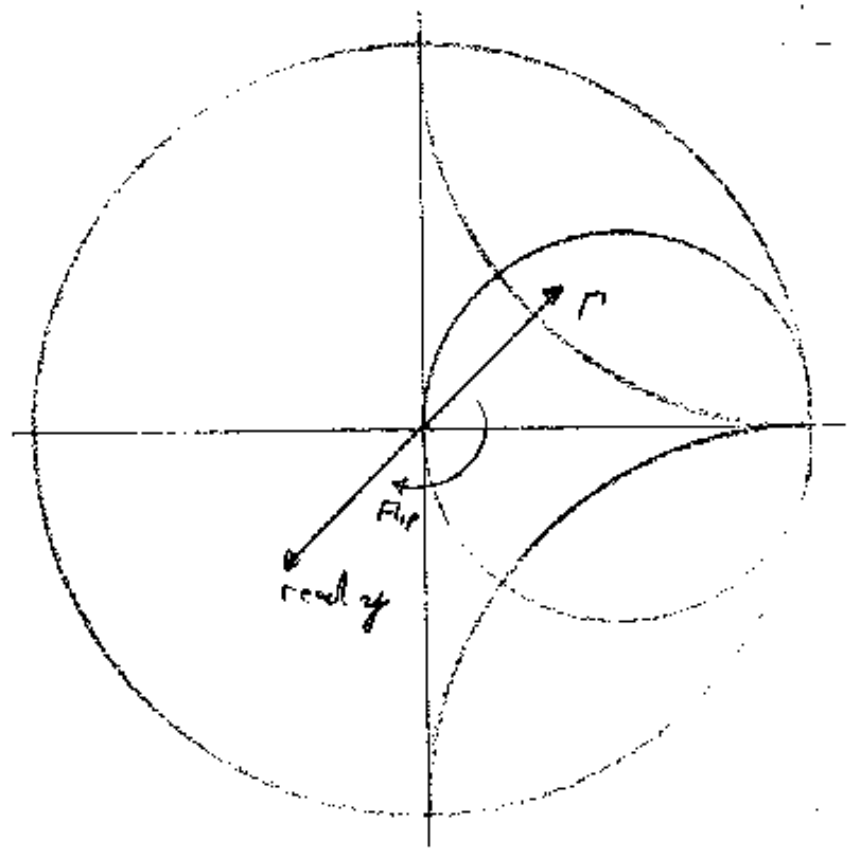
If $y = 1 + j1$ what is Γ?

a) plot $1 + j1$ on chart

Vector $= .445 \angle 64°$

b) flip vector by 180°

$\Gamma = .445 \angle -116°$



Example 2

If $\Gamma = 0.5\angle 45°$ $\quad Z_o = 50\Omega$

What is Y

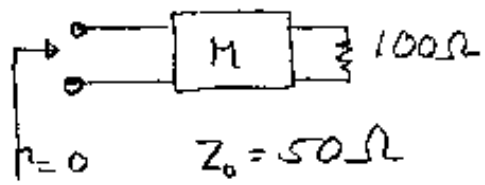
a) Plot Γ

b) Flip Γ by $180°$

c) read coordinate $= .38 - j.36$

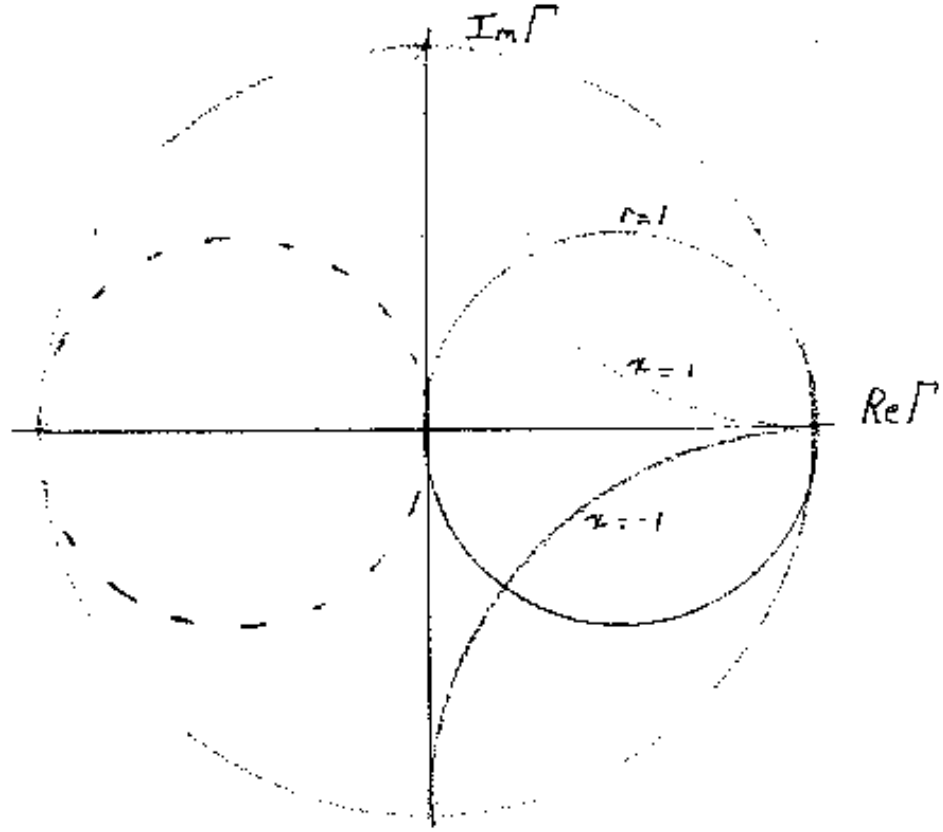
d) $Y = \frac{1}{50\Omega}(.38 - j.36) = (7.6 - j7.2)\times 10^{-3}$ mhos

Matching Example: Match a 100Ω load to a 50Ω transmission line system @ 100 MHz



We want to use only lossless elements such as capacitors & inductors so we don't dissapate any power in the matching network.

To make things easier, let's draw in an extra circle on the Smith Chart, the mirror image of the $r=1$ circle

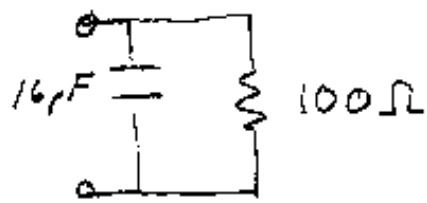


1) We need to go from $z = 2 + j0$ to $z = 1 + j0$ on Smith chart. We won't get any closer by adding a series impedance to the load. So we need to Flip over to admittance which is at $0.5 + 0j$.

2) Now add positive imaginary admittance to the load until we reach the the mirror image circle

$jb = j.5$

$j(.5)\left(\frac{1}{50\Omega}\right) = 2\pi\, 100MHz\, C$



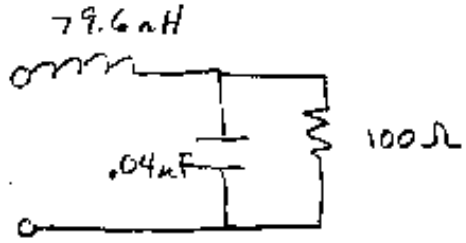
$C = 16\,pF$

3) Now Flip back to impedance
We now land on the $r=1$ circle at $x=-1$

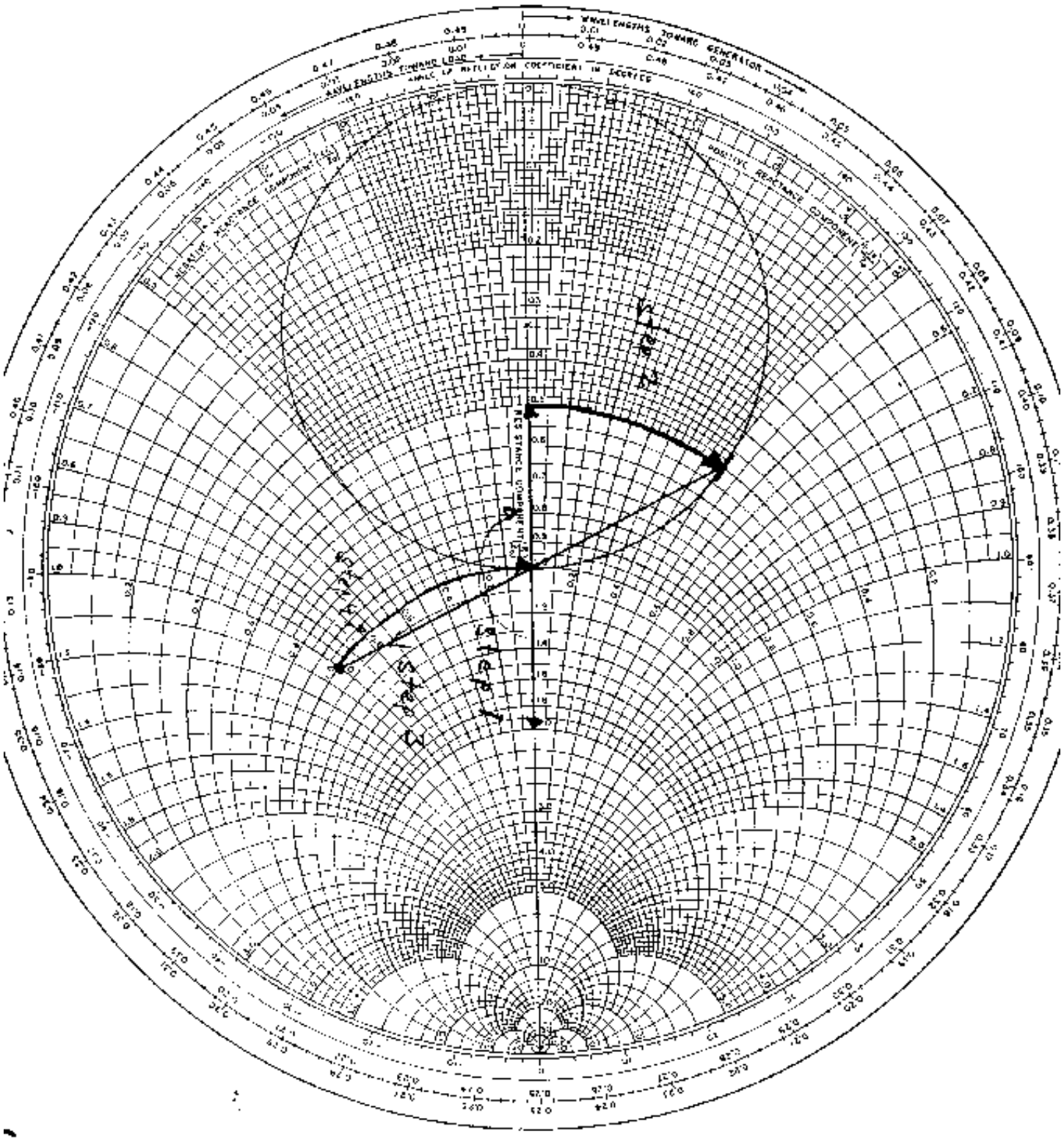
4) Add positive imaginary admittance to get us to $z = 1 + j0$

$jx = j1$

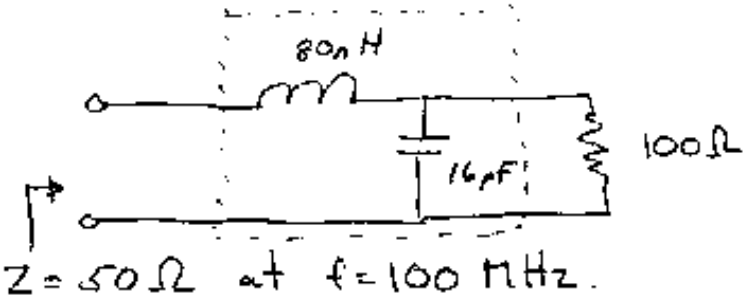
$(j1)\,50\Omega = 2\pi\, 100MHz\, L$



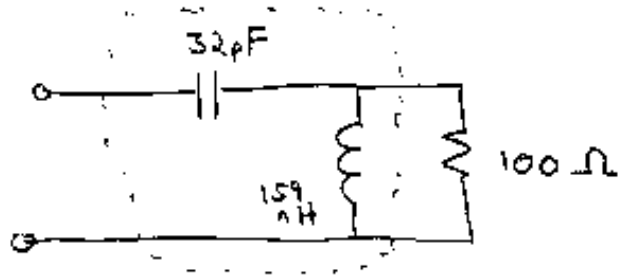
$L = 79.6\,nH$


WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
POSITIVE REACTANCE COMPONENT
NEGATIVE REACTANCE COMPONENT
RESISTANCE COMPONENT

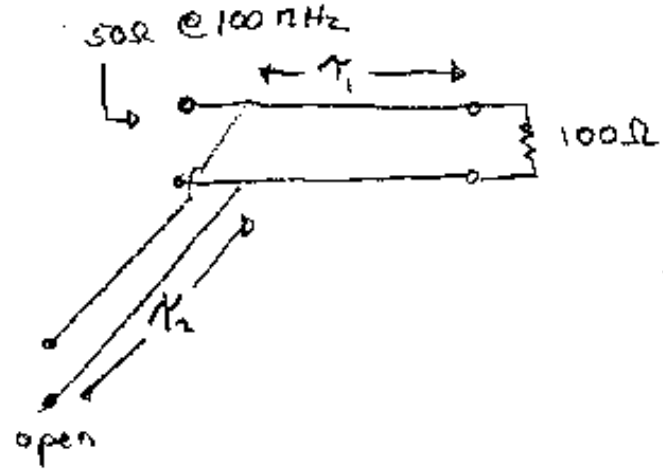
Final Answer



This answer would have also worked:



Single stub tuner



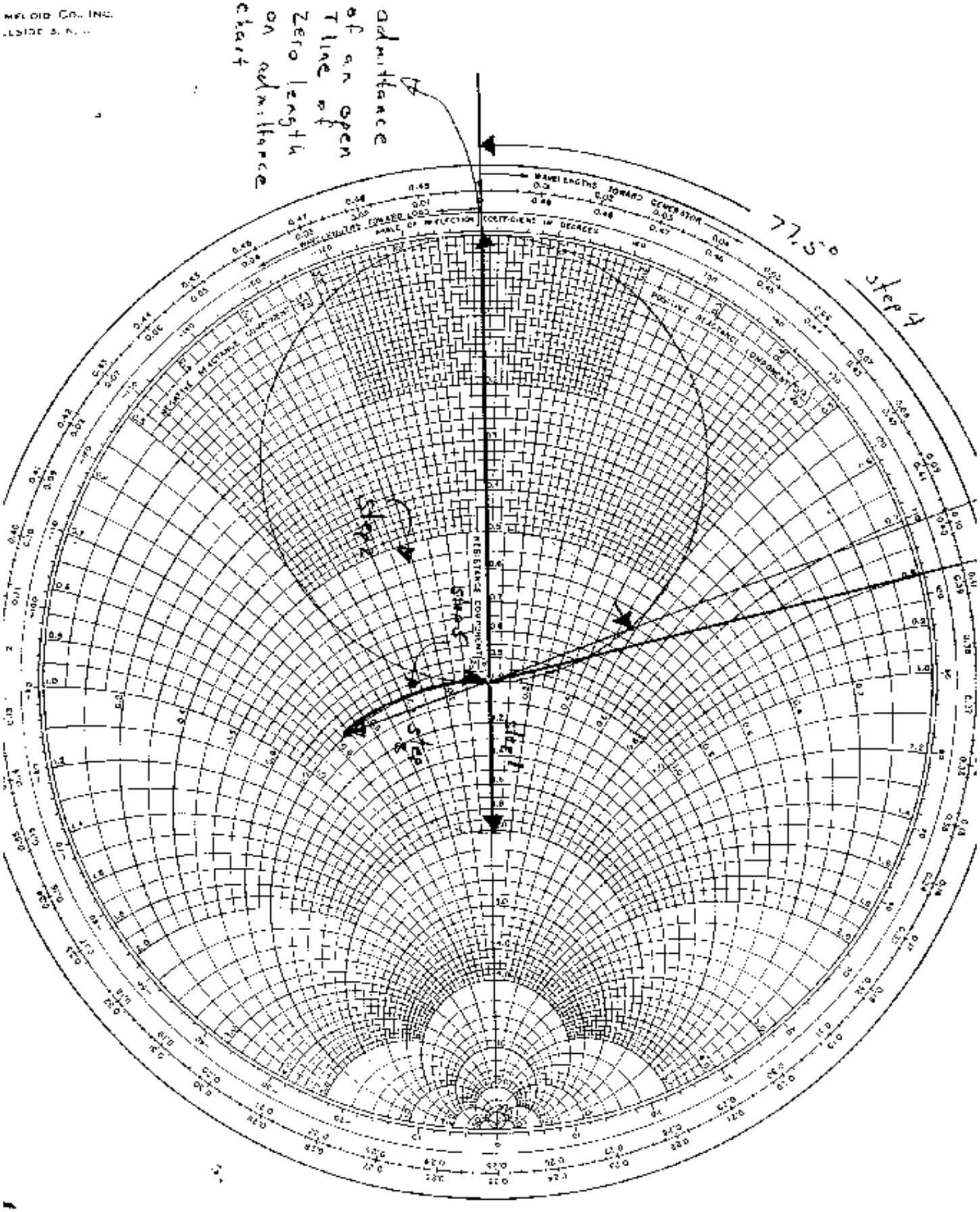
1) plot reflection coefficient

2) Rotate reflection coefficient by traveling an angle of $\Theta = 251°$ towards the generator

$$\Theta = 2 \cdot 360° \cdot 100\,MHz \cdot \tau_1$$

$\tau_1 = 3.5\,nS$ to land on mirror image circle

MELOID CO., INC.
LESIDE, S. N. ...



Philip H. Smith "Transmission Line Calculator" Electronics Jan 1939 and Jan 1944

Printed in

3) Flip over by 180° for admittance
At this point we have an normalized admittance of
$1 - j.8$ admittance

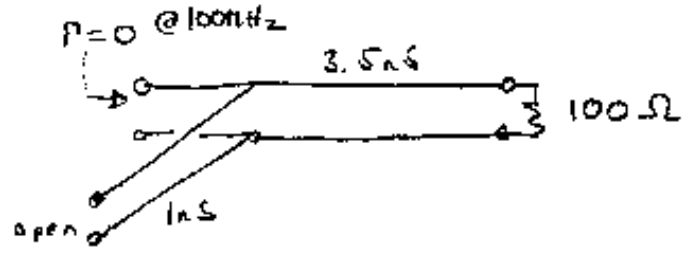
We need to add an normalized admittance of $+ j.8$ to transmission line

4. An open circuit line of zero length has an admittance of $j0$. If we rotate by 77.5° the line will have an admittance of

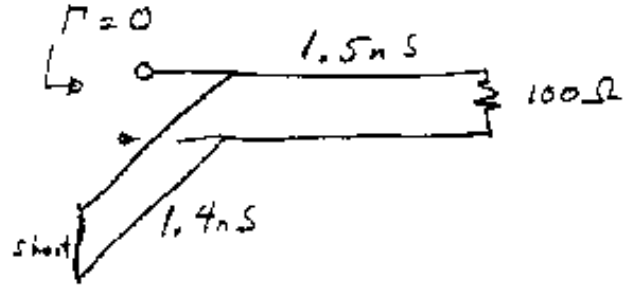
$$77.5° = 2 \cdot 360° \cdot 100MHz\ \tau_2$$

$$\tau_2 = 1nS$$

5.

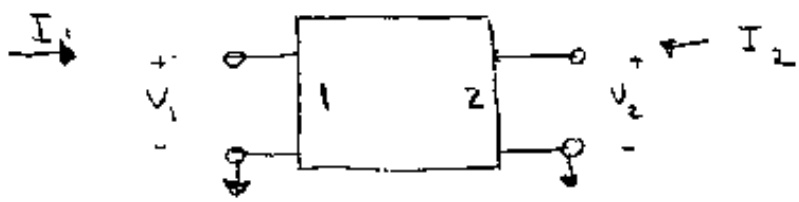


This answer would have worked as well



<u>Scattering Parameters</u> (S parameters)

We have only worried about reflection so far. What about transmission? Consider a device which has two ports (or spigots)



The device can be characterized by 2X2 matrix.

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

$$[V] = [Z][I]$$

but since $V_i = V_i^+ + V_i^-$

$$Z_0 I_i = V_i^+ - V_i^-$$

where $V_i^\pm$ stand for forward and reverse waves travelling into port i.

We can characterize the Circuit by the voltage waves

$$V_1^- = S_{11} V_1^+ + S_{12} V_2^+$$

$$V_2^- = S_{21} V_1^+ + S_{22} V_2^+$$

$$[V^-] = [S][V^+]$$

The S matrix is called the scattering matrix. This is actually the <u>un-normalized</u> scattering matrix.

For convenience define

$$a_i = \frac{V_i^+}{\sqrt{2Z_{o_i}}}$$

$$b_i = \frac{V_i^-}{\sqrt{2Z_{o_i}}}$$

where Z_{o_i} is the characteristic impedance of the transmission line connecting port i.

$\therefore$ $|a_i|^2$ = forward power into port i

$|b_i|^2$ = reflected power out of port i

$$b_1 = s_{11} a_1 + s_{12} a_2$$
$$b_2 = s_{21} a_1 + s_{22} a_2$$

$$[b] = [s][a]$$

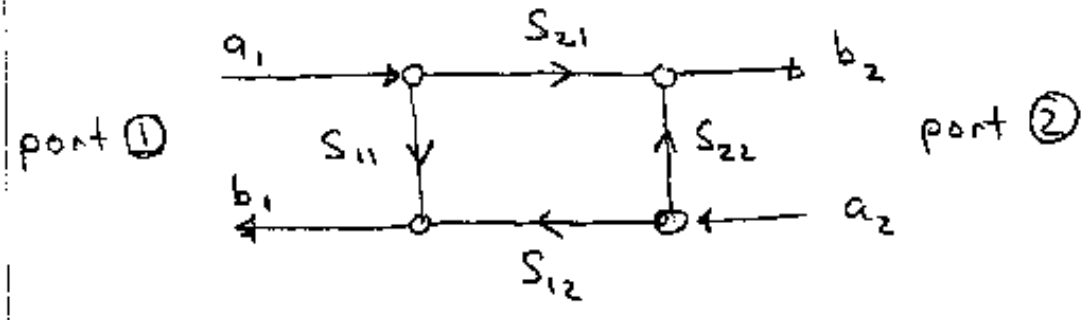
where $[s]$ is the power normalized scattering matrix.

where

$$s_{ij} = \sqrt{\frac{Z_{0j}}{Z_{0i}}} \; S_{ij}$$

Normalized s parameters are most commonly used. i.e. $Z_{01} = Z_{02}$

The s parameters can be drawn pictorially



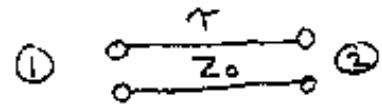
S_{11} and S_{22} can be thought of as reflection coefficients

S_{21} and S_{12} can be thought of as transmission coefficients

Remember, the S parameters are complex numbers where the angle corresponds to a phase shift between the input & output waves.

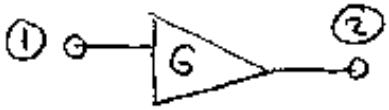
<u>Some common S matrices</u>

Transmission line



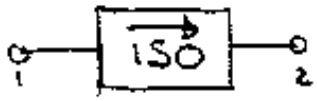
$$[S] = \begin{bmatrix} 0 & e^{-j360^\circ f\tau} \\ e^{-j360^\circ f\tau} & 0 \end{bmatrix}$$

An amplifier



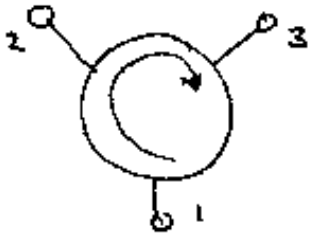
$$S = \begin{bmatrix} 0 & 0 \\ G & 0 \end{bmatrix}$$

An isolator



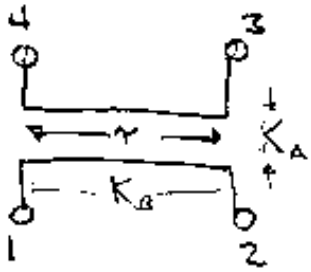
$$S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

A circulator



$$S = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

A directional coupler



$$S = \begin{bmatrix} 0 & K_B e^{-j2\pi f\tau} & jK_A e^{-j2\pi f\tau} & 0 \\ K_B e^{-j2\pi f\tau} & 0 & 0 & jK_A e^{-j2\pi f\tau} \\ jK_A e^{-2\pi f\tau} & 0 & 0 & K_B e^{-j2\pi f\tau} \\ 0 & jK_A e^{-j2\pi f\tau} & K_B e^{-j2\pi f\tau} & 0 \end{bmatrix}$$

Two Useful Laws

A. Lorentz reciprocity

If the device is made out of linear materials ie. resistors, inductors, capacitors metal etc.

Then $[S]$ is symmetric

$$[S]_{\text{Transpose}} = [S]$$

or $$S_{ij} = S_{ji} \quad \text{for } j \neq i$$

This is the same as saying that the transmitting pattern of an antenna is the same as the receiving pattern of the attenna.

From the above examples

Reciprocal Devices $\left\{ \begin{array}{l} \text{Transmission line} \\ \text{Directional Coupler} \end{array} \right.$

Non-reciprocal Devices $\left\{ \begin{array}{l} \text{Amplifier} \\ \text{Isolator} \\ \text{Circulator} \end{array} \right.$

B. Lossless Devices

The S parameter matrix of a lossless device is <u>Unitary</u>

$$[S^*]_{\text{transpose}} [S] = [I]$$

To check Loss, Sum up the magnitudes of any row or column, the sum must be 1.

$$1 = \sum_j |S_{ij}|^2 \quad \text{for all } i$$

or

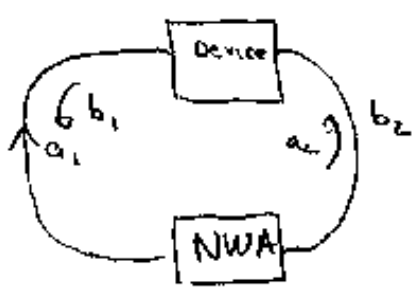
$$1 = \sum_i |S_{ij}|^2 \quad \text{for all } j$$

for the device to be lossless.

Lossless devices from above examples
- Transmission line
- Isolator
- Circulator
- Direction coupler if $|K_A|^2 + |K_B|^2 = 1$

Network Analyzers

Network analyzers measure the s parameters as a function of frequency



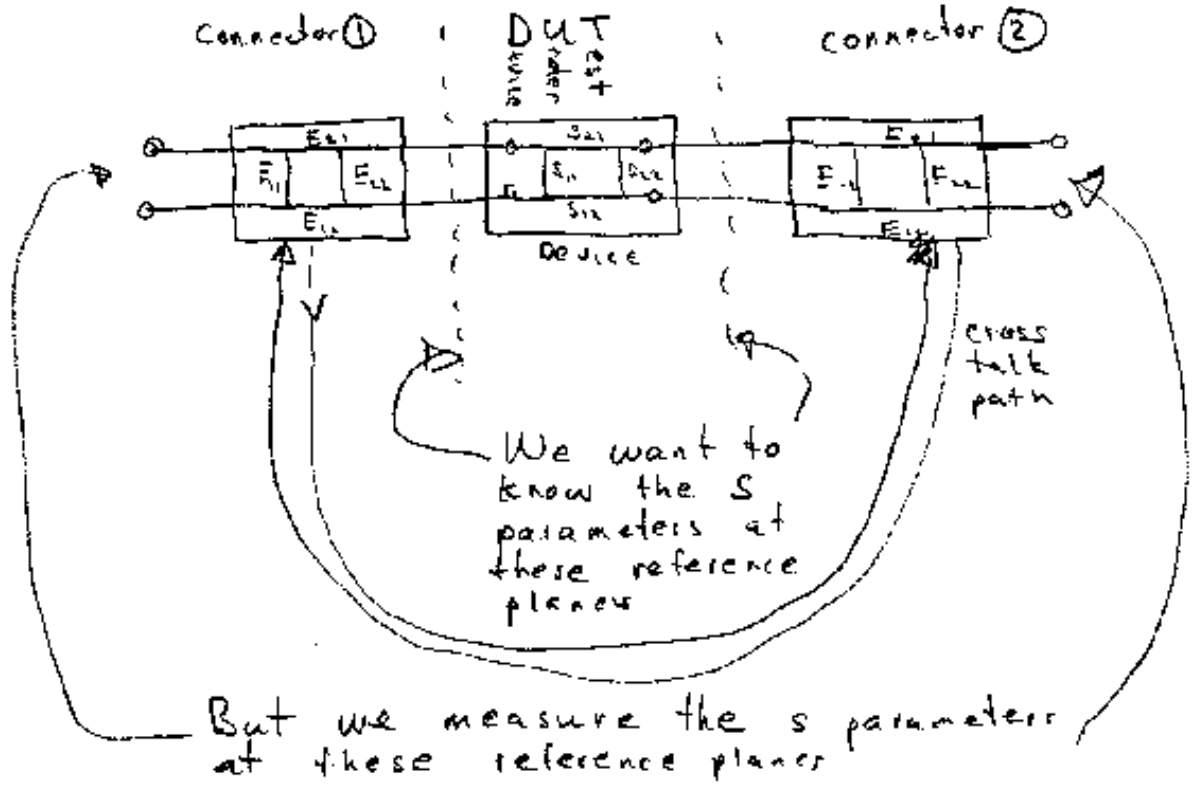
$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} \qquad S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0}$$

To make $a_2 = 0$ the NWA has to be matched at port 2 because $b_2 \neq 0$ for $S_{21} \neq 0$

$$S_{22} = \frac{b_2}{a_2}\Big|_{a_1=0} \qquad S_{12} = \frac{b_1}{a_2}\Big|_{a_1=0}$$

Calibration

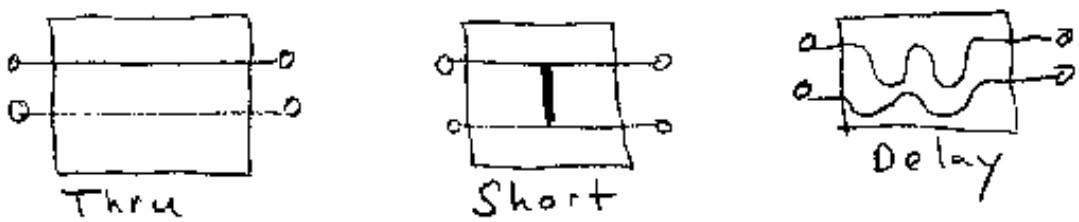
To measure the pure S parameters of a device we need to eliminate the effects the cables, connectors, etc attaching the device to the network analyzer



There are 10 unknowns in the connectors
We need 10 independent measurements to eliminate these unknowns.

Come up with calibration standards.
We place the standards in where the DUT would be and measure the S-parameters of the standards. Since we know apriori the S parameters of the standards, we can

determine the unknown S parameters of the connectors. Once these connector parameters are known, they can be mathematically eliminated from the measurements once the DUT is placed back in the measuring fixture. Since we get to measure 4 S parameters for each calibration standard, we need at least 3 independent standards. One possible set is



This is known as calibration or de-embedding.

Group Velocity and phase velocity

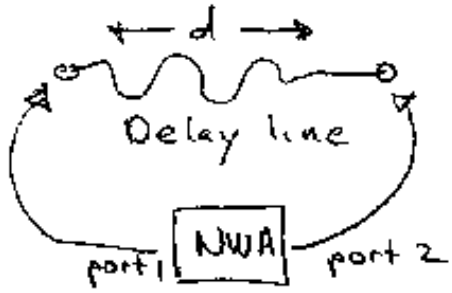
A pure sine wave can be written as

$$V = V_0 \, e^{j(\omega t - \beta z)}$$

The phase velocity is equal to

$$v_{phase} = \frac{\omega}{\beta}$$

We can measure the phase velocity by setting up an input wave to delay line and measuring the phase difference between the input and output wave.



We measure the angle of S_{21} at a given freq

$$\arg(S_{21}) = \beta d = \Theta$$

$$v_{phase} = \frac{\text{freq} \cdot d}{\Theta / 360^\circ}$$

This is the phase velocity of a single wave. Suppose we wish to measure the velocity of a pulse thru a device.

The pulse is composed of many frequencies. Each of the frequencies might be travelling at a different phase velocity. The group velocity is

$$v_{group} = \frac{\partial \omega}{\partial \beta}$$

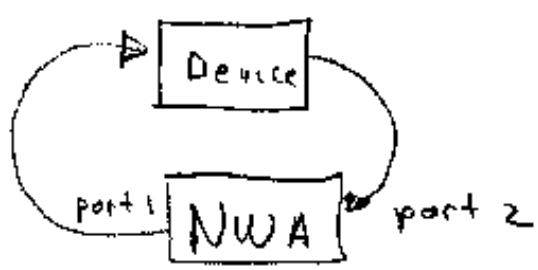
The group delay is the time it takes the pulse to travel a distance d

$$\tau_{group} = \frac{d}{v_{group}}$$

$$\tau_{group} = d \frac{\partial \beta}{\partial \omega}$$

but βd is the phase of the wave

$$\tau_{group} = \frac{\partial \theta}{\partial \omega}$$

$$\tau_{group} = \frac{1}{360^\circ} \frac{\partial \theta}{\partial f}$$

To measure the group delay of a device, Measure S21